Signal Processing and Machine Learning on Graphs

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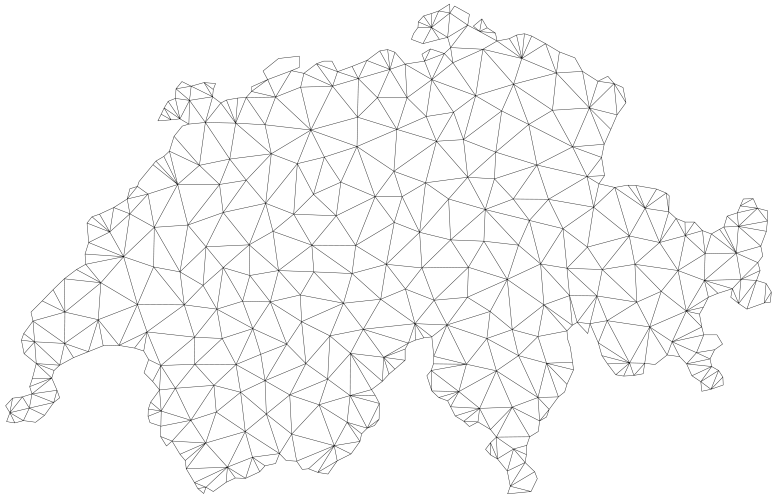
Summer School in Economic Networks
Oxford, June 2019

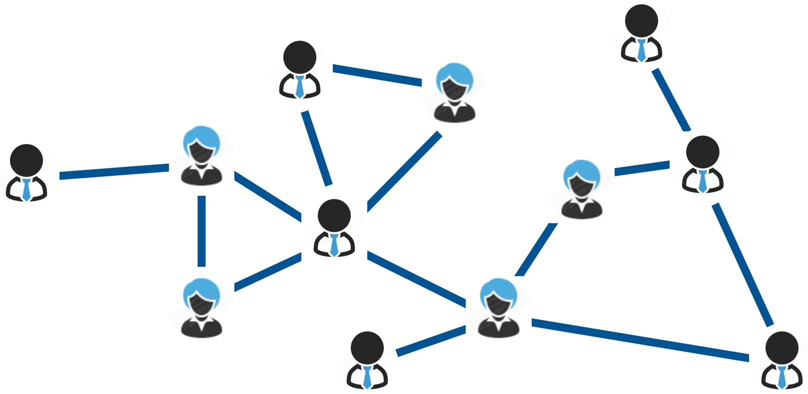
UNIVERSITY OF OXFORD

Networks are everywhere

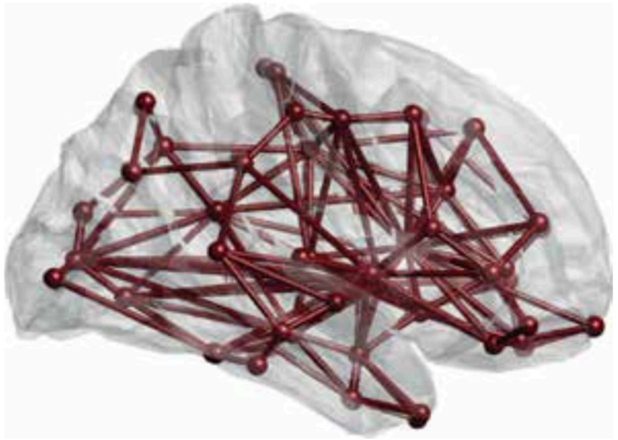


geographical network

traffic network

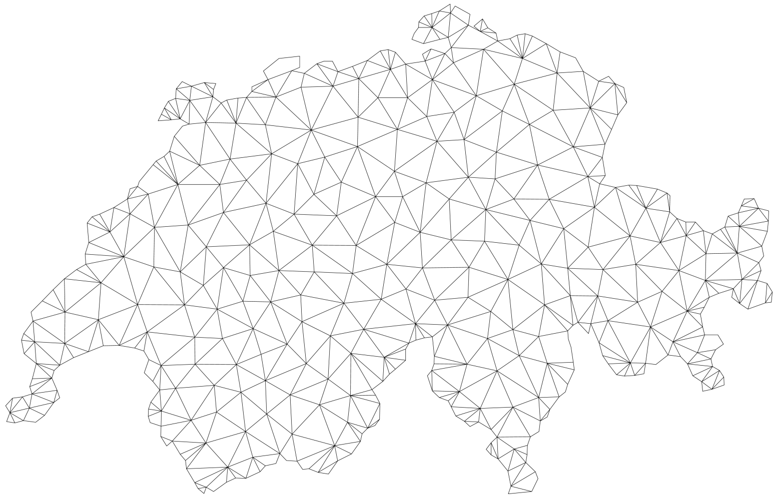


social network



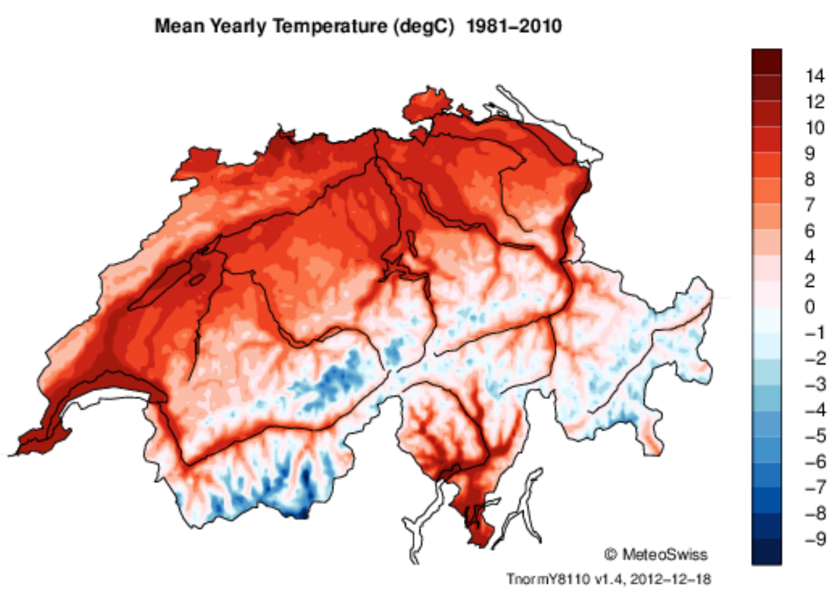
brain network [Huang18]

Network-structured data are everywhere



- vertices
  - geographical regions
- edges
  - geographical proximity between regions

Network-structured data are everywhere

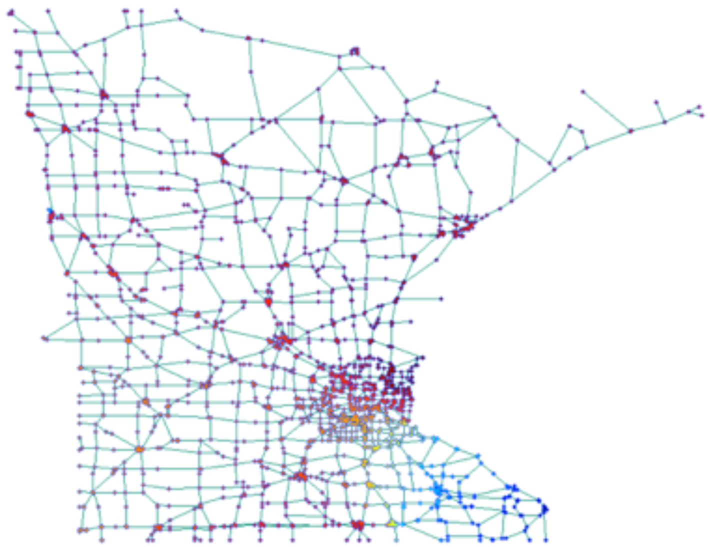


- vertices
  - geographical regions
- edges
  - geographical proximity between regions
- signal
  - temperature records in these regions

Network-structured data are everywhere

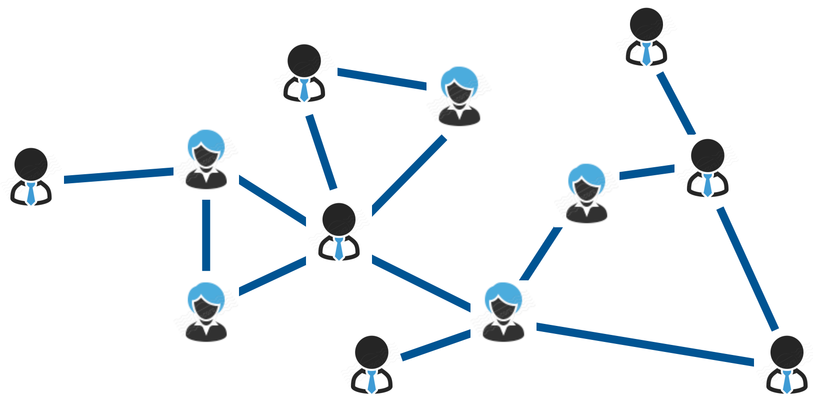
- vertices
  - road junctions
- edges
  - road network

Network-structured data are everywhere



- vertices
  - road junctions
- edges
  - road network
- signal
  - traffic congestion at junctions

Network-structured data are everywhere



- vertices
  - individuals
- edges
  - friendship between individuals

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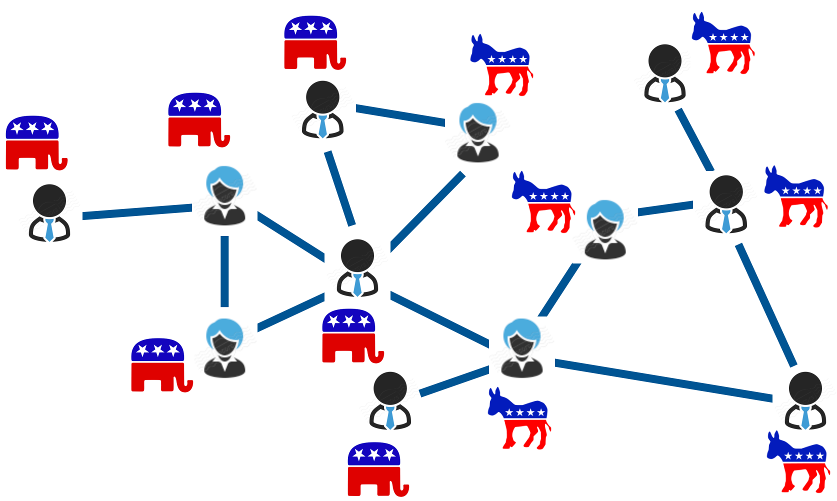
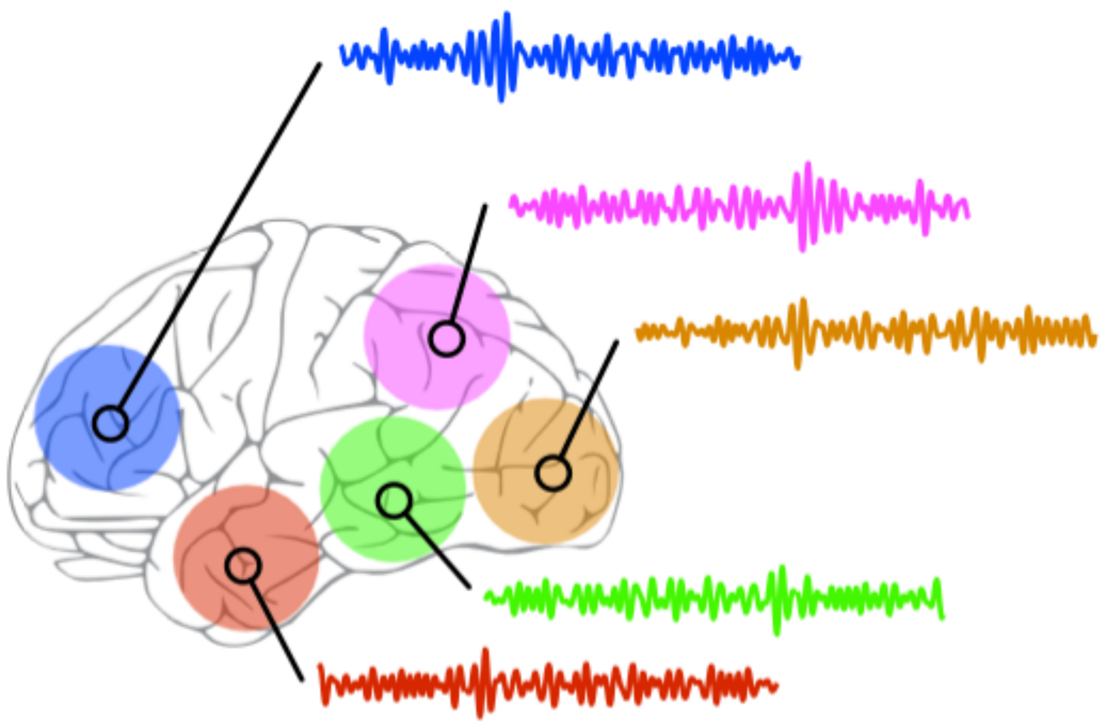


- vertices
  - individuals
- edges
  - friendship between individuals
- signal
  - political view

Network-structured data are everywhere

- vertices
  - brain regions
- edges
  - structural connectivity between brain regions

Network-structured data are everywhere



- vertices
  - brain regions
- edges
  - structural connectivity between brain regions
- signal
  - blood-oxygen-level-dependent (BOLD) time series

Richiardi et al., "Machine learning with brain graphs," IEEE SPM, 2013.

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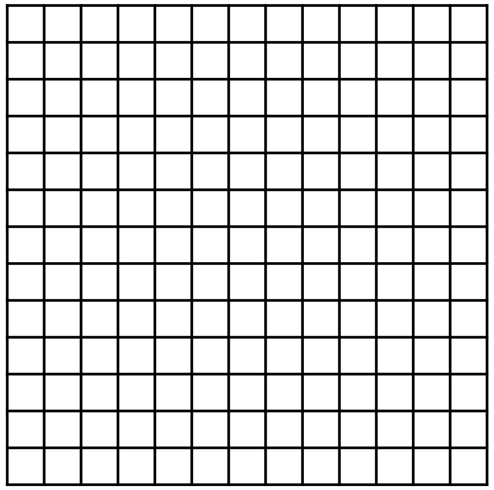
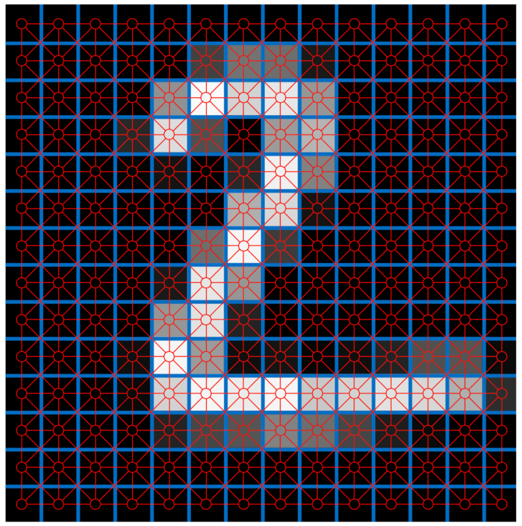


- vertices
  - pixels
- edges
  - spatial proximity between pixels

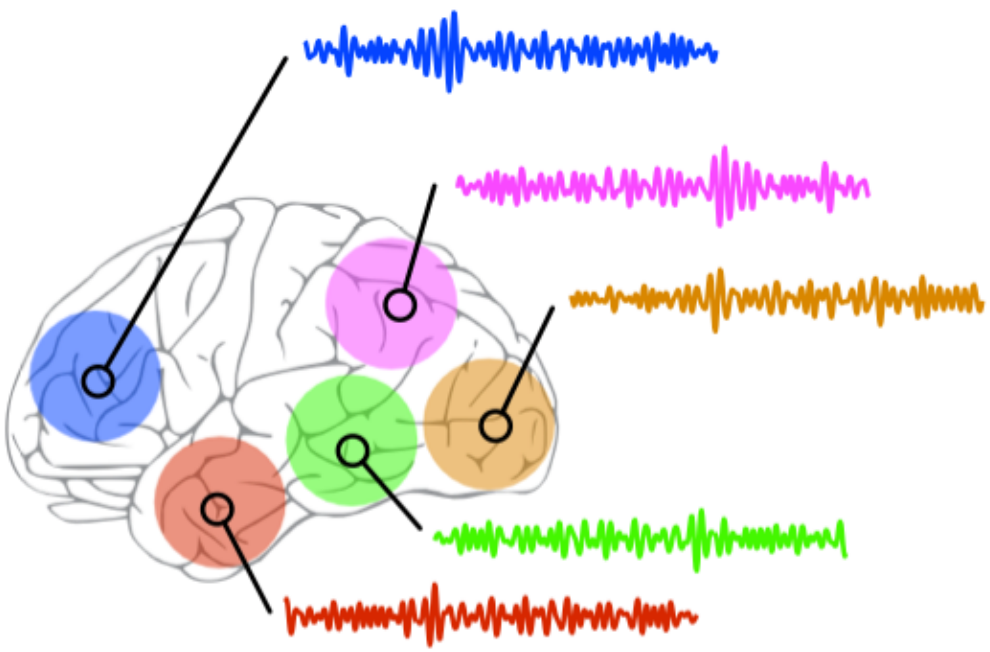
Network-structured data are everywhere



- vertices
  - pixels
- edges
  - spatial proximity between pixels
- signal
  - pixel values

Monti et al., "Geometric deep learning on graphs and manifolds using mixture model CNNs," CVPR, 2017.

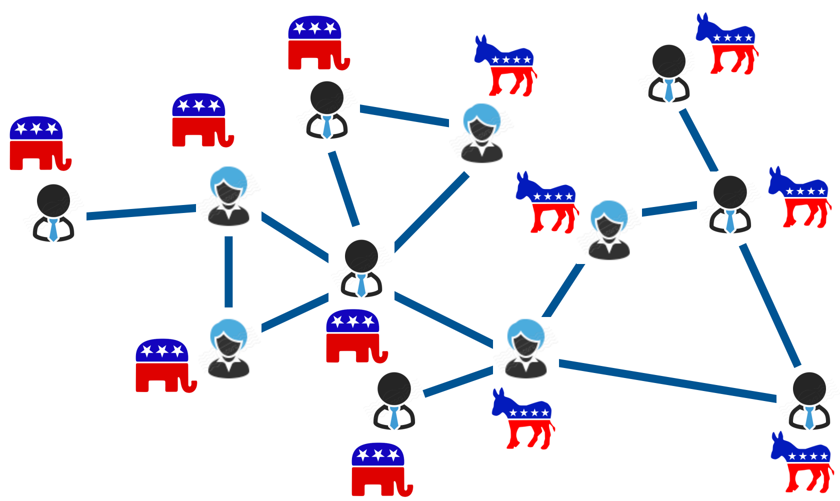
Learning with network-structured data



medical condition?
no medical condition?

network-wise classification

Learning with network-structured data



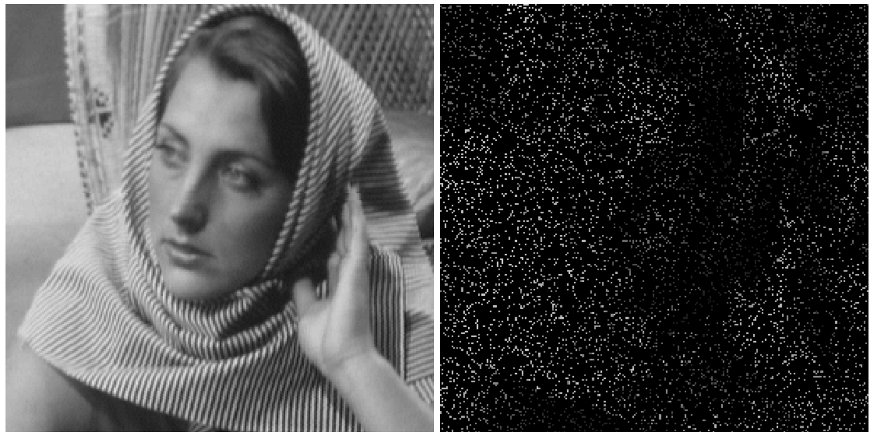
democratic?
republican?

vertex-wise classification

Learning with network-structured data

denoising and inpainting (inspired by image processing)

Learning with network-structured data



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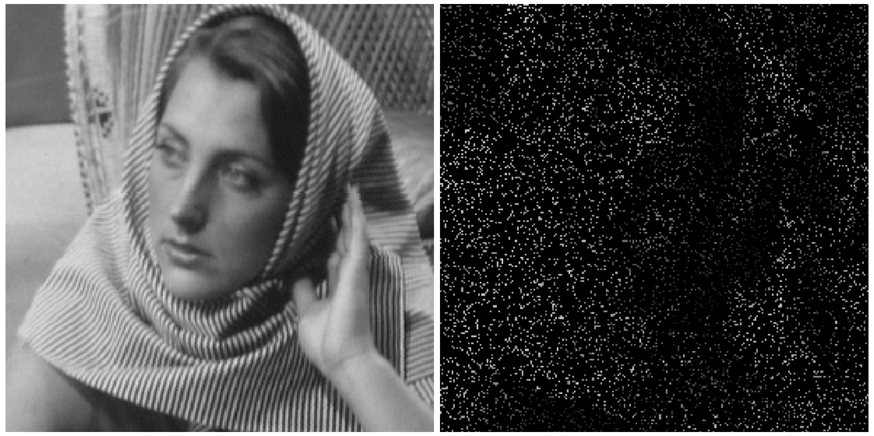

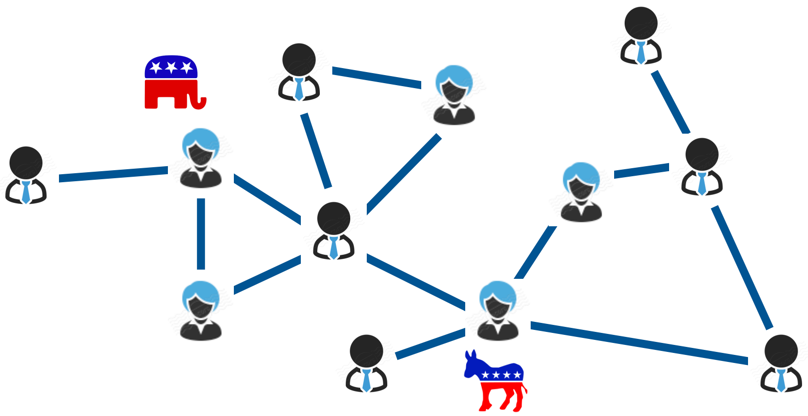

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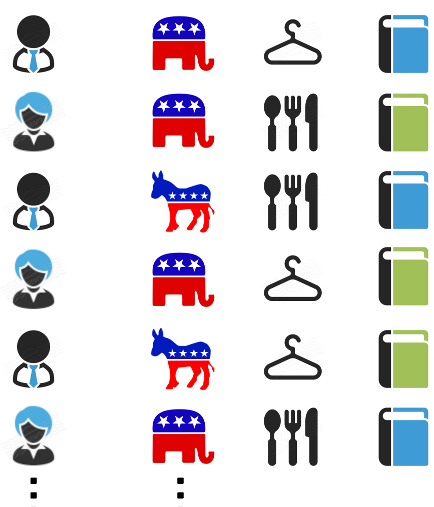
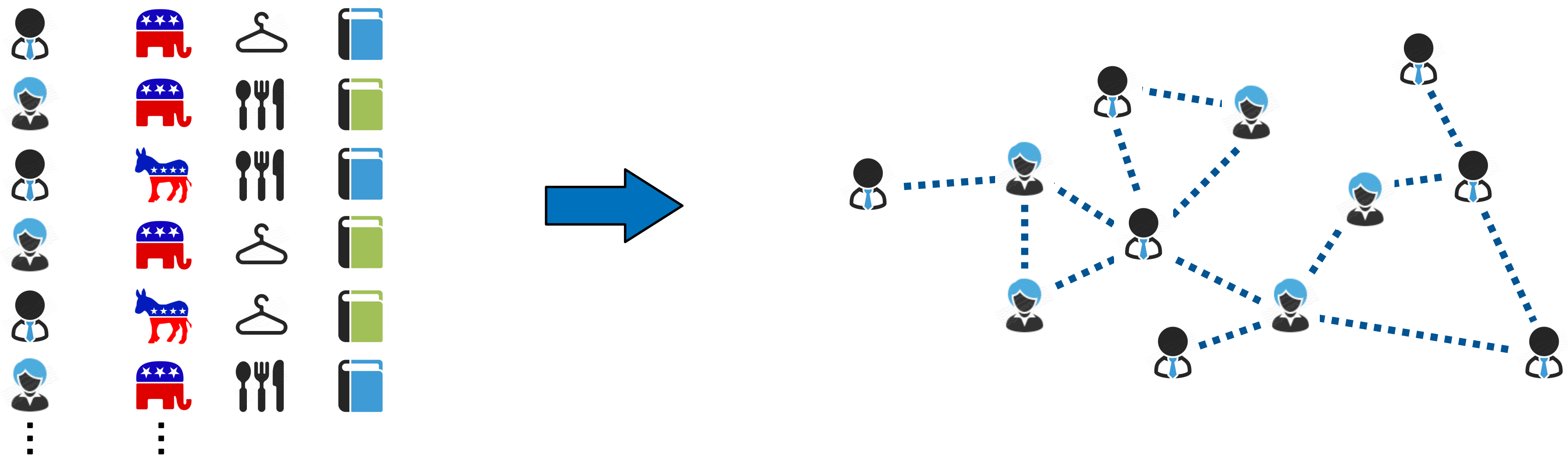


inferring network structure from data

Learning with network-structured data



inferring network structure from data

How to incorporate network into learning?

- Straightforward approach: embed the network into a Euclidean space

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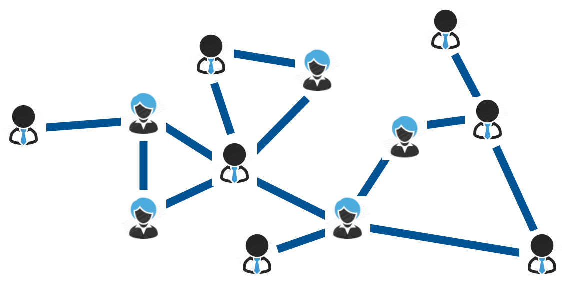

network

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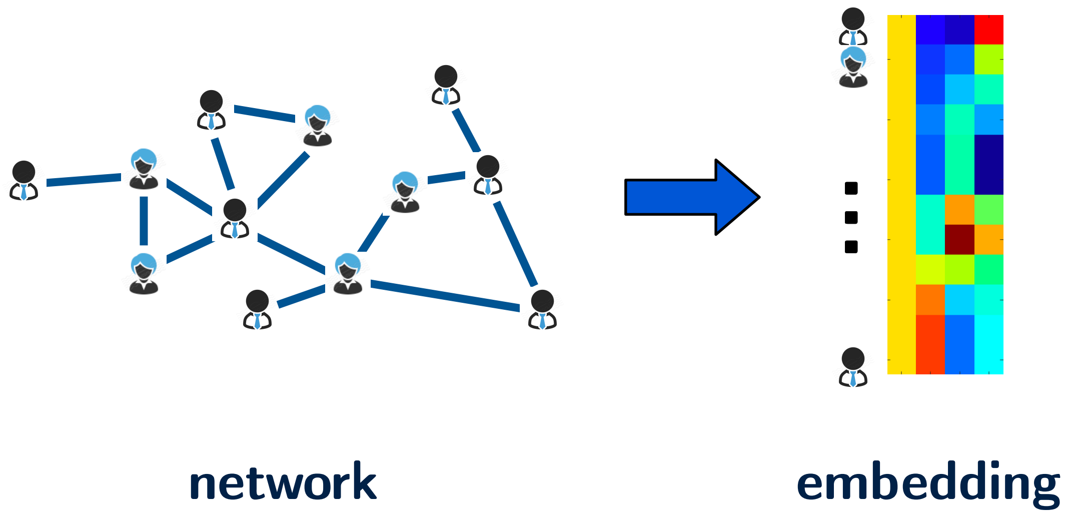

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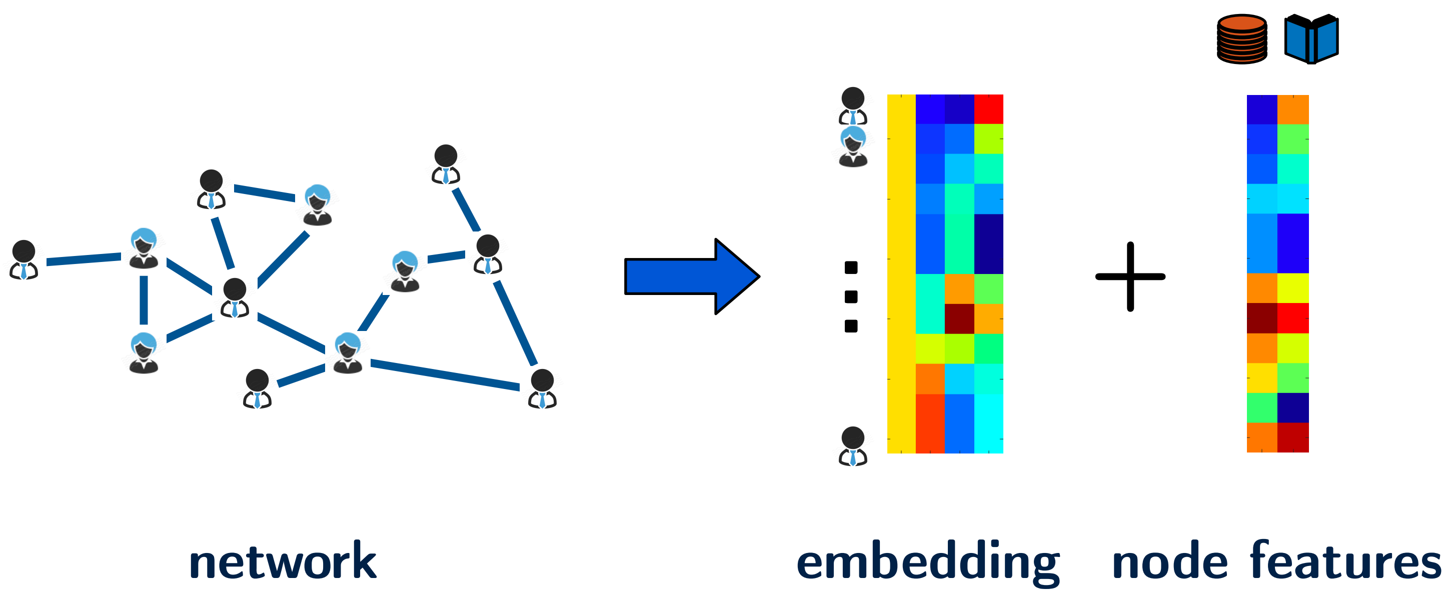

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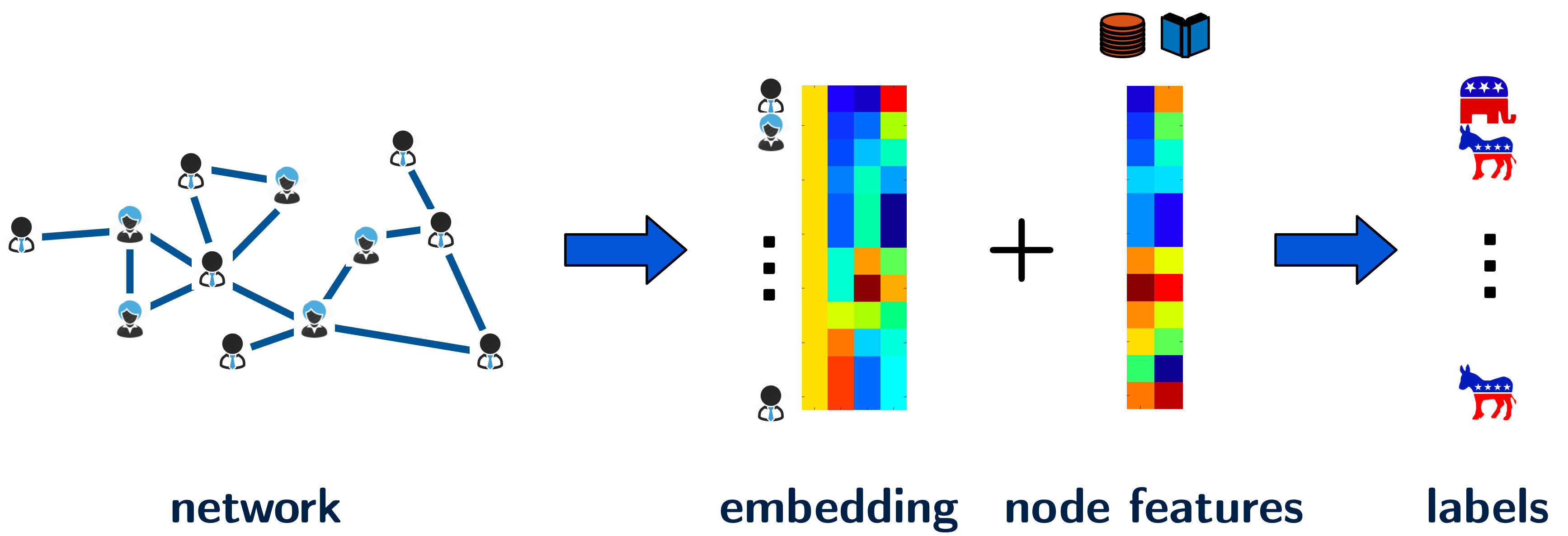

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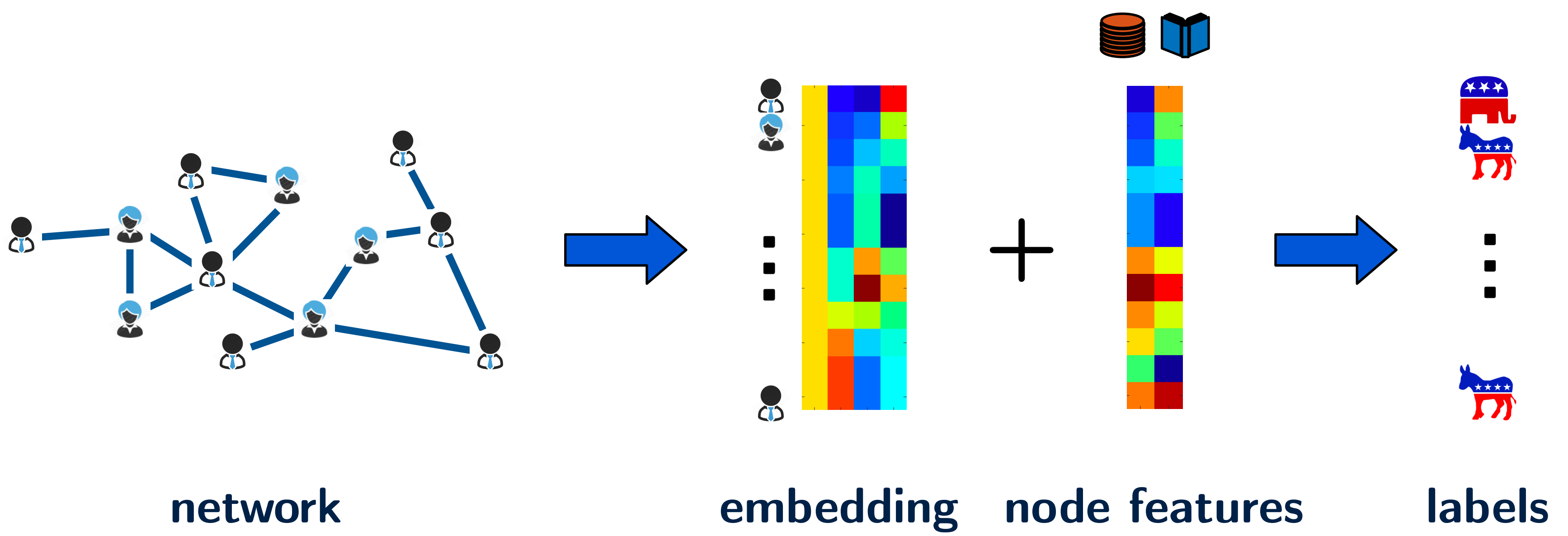

- embedding of network structure leads to information loss
- need for new models & tools that directly incorporate structure in data analysis

Outline

- Graph signal processing (GSP): Basic concepts
- Spectral filtering: Basic tools of GSP
- Connection with literature
- Research topics inspired by GSP
- Applications

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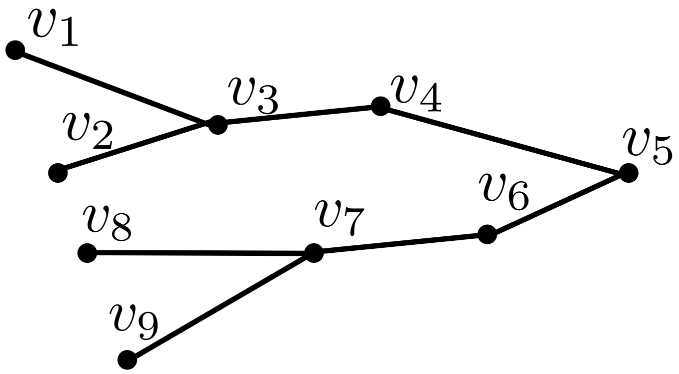

Graph signal processing

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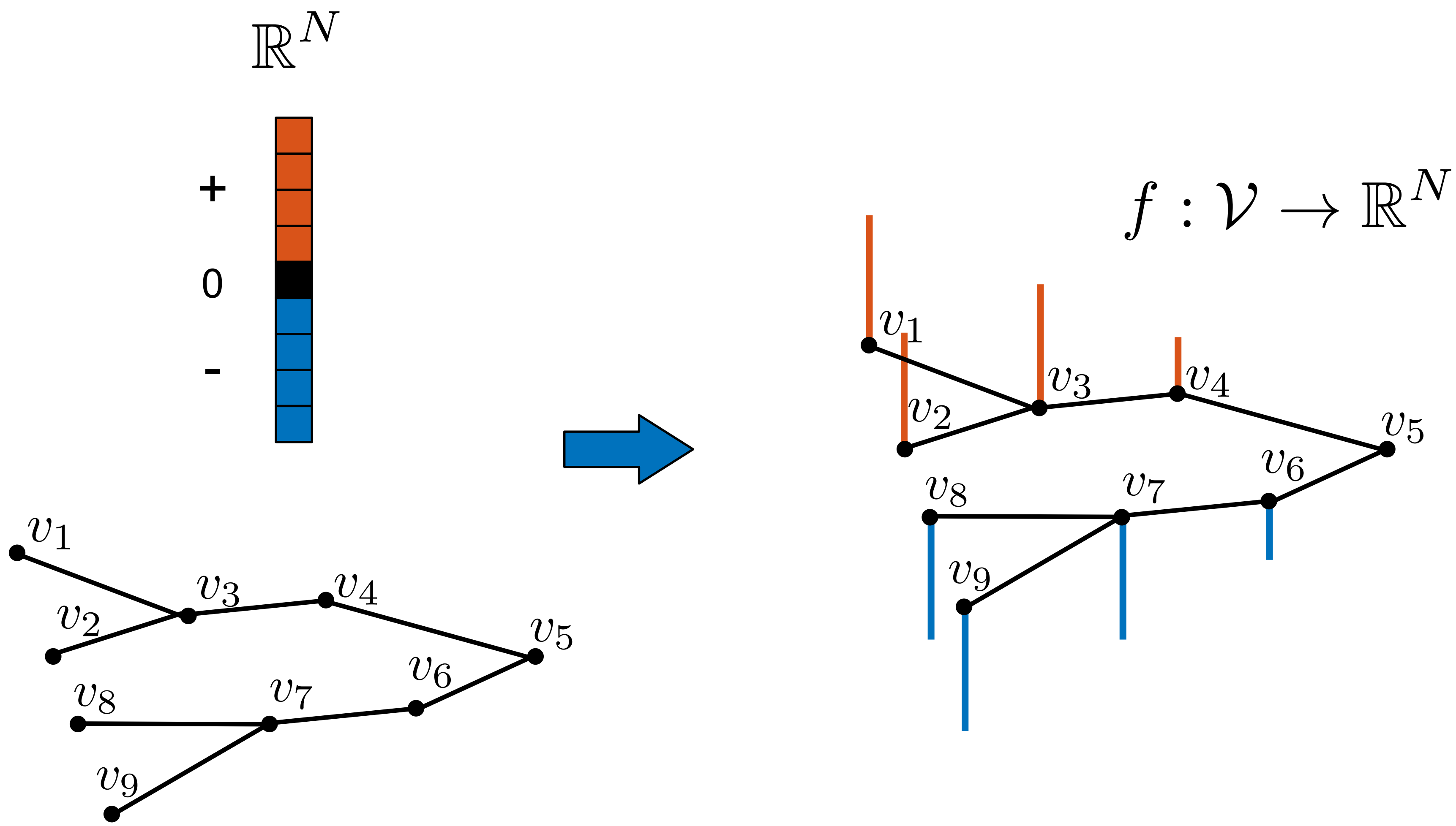

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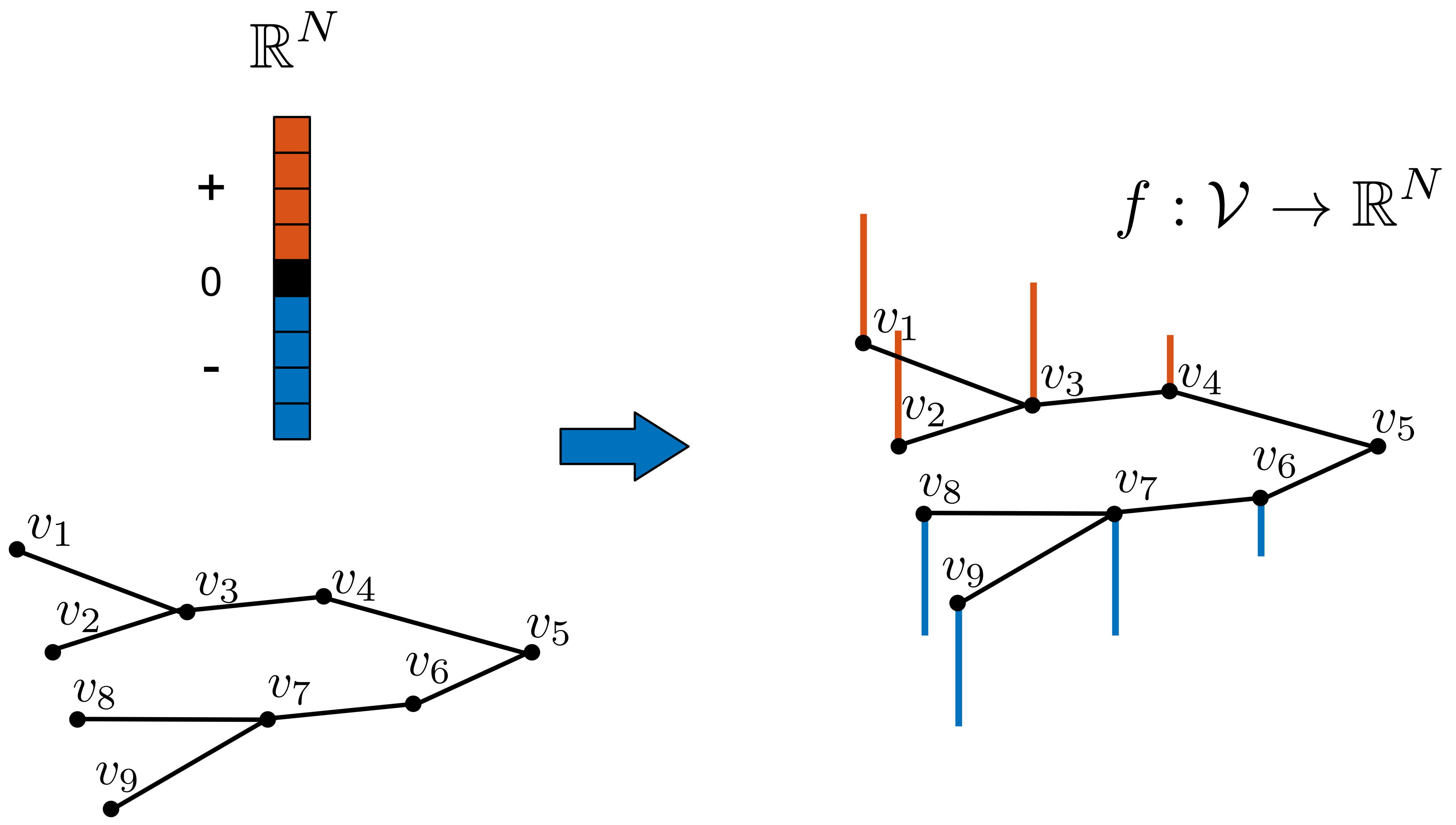
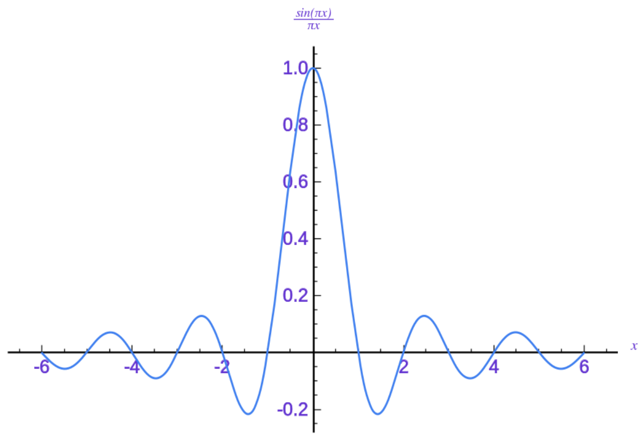
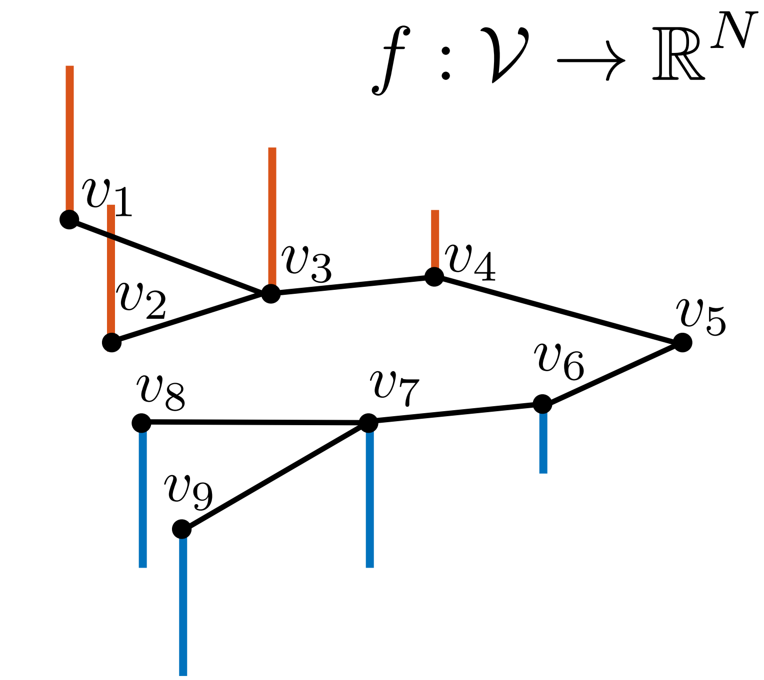


takes into account both structure (edges) and data (values at vertices)

Graph signal processing





how to generalise classical signal processing tools on irregular domains such as graphs?

Graph signal processing

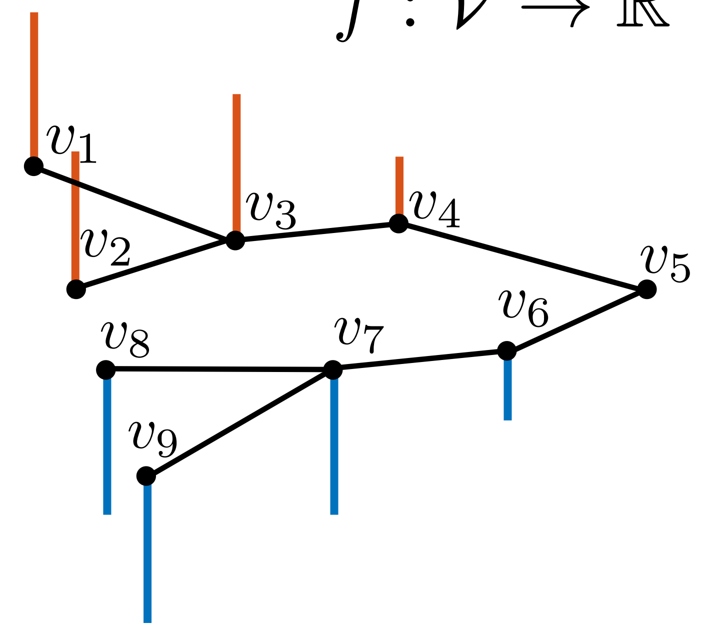
- Graph signals provide a nice compact format to encode structure within data

- Generalisation of classical signal processing tools can greatly benefit analysis of such data

- Numerous applications: Transportation, biomedical, social, economic network analysis

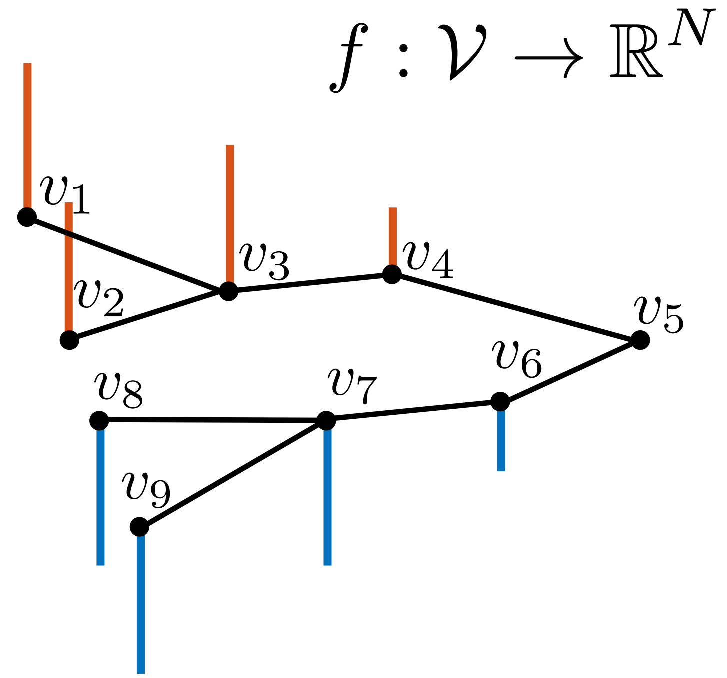
- An increasingly rich literature
  - classical signal processing
  - algebraic and spectral graph theory
  - computational harmonic analysis
  - machine learning

$f : \mathcal{V} \to \mathbb{R}^N$



Two paradigms

- Main GSP approaches can be categorised into two families:
  - vertex (spatial) domain designs
  - frequency (graph spectral) domain designs



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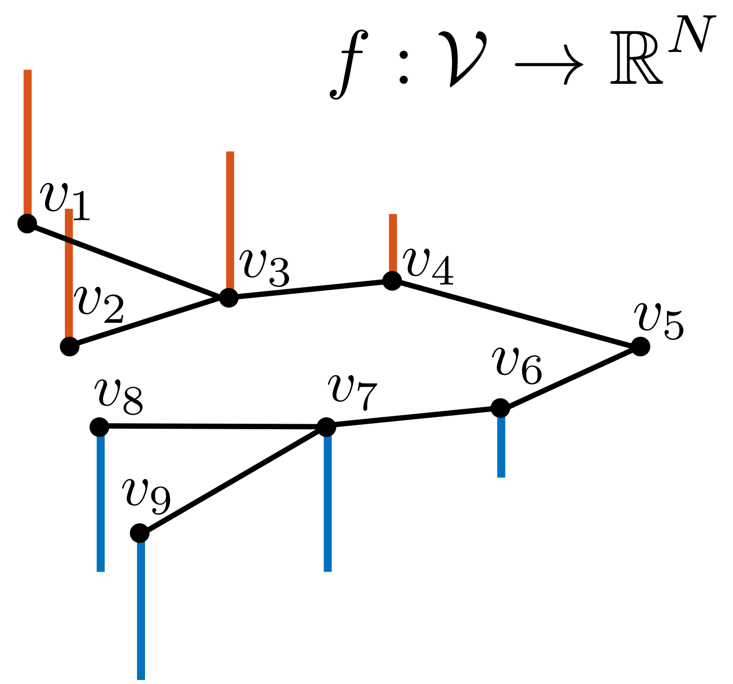
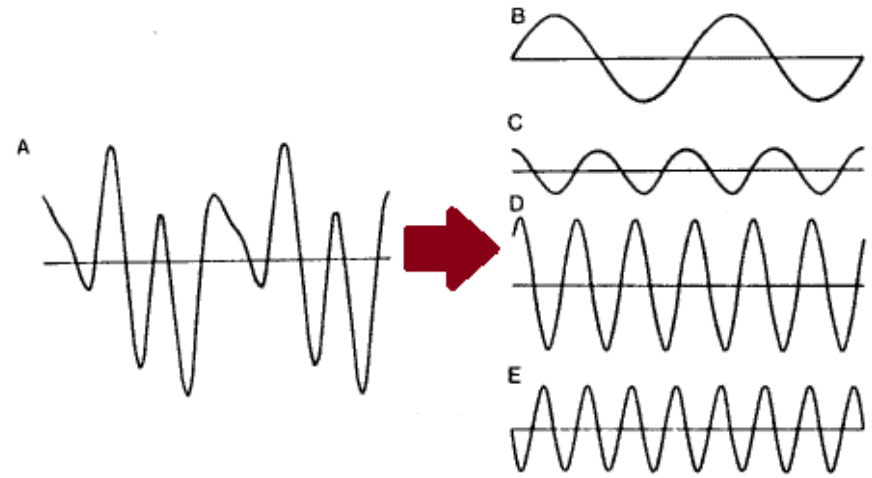
important for analysis of signal properties



Need for notion of frequency

- Classical Fourier transform provides frequency domain representation of signals



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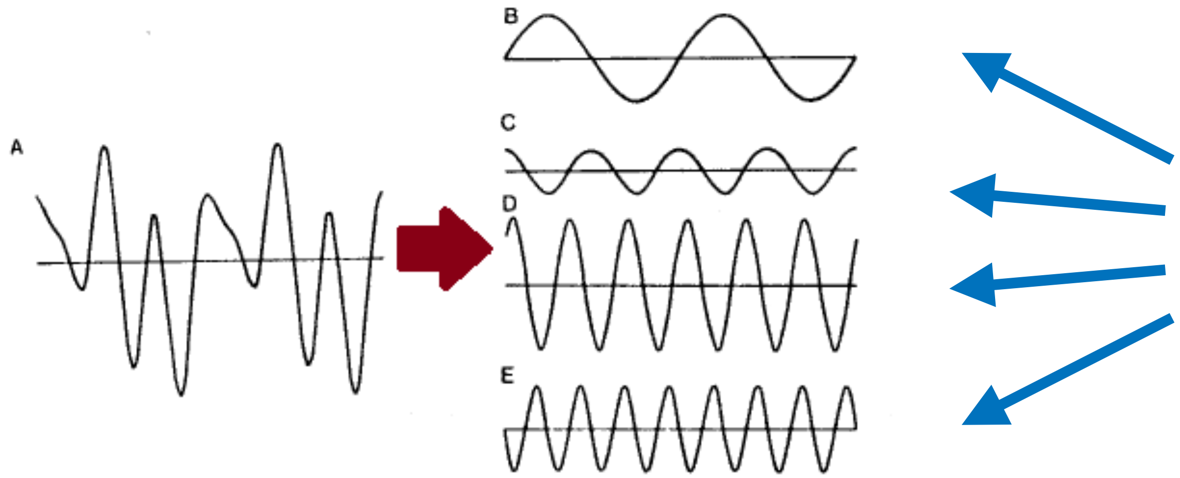

- "building blocks" of signal
- different frequency (oscillation)

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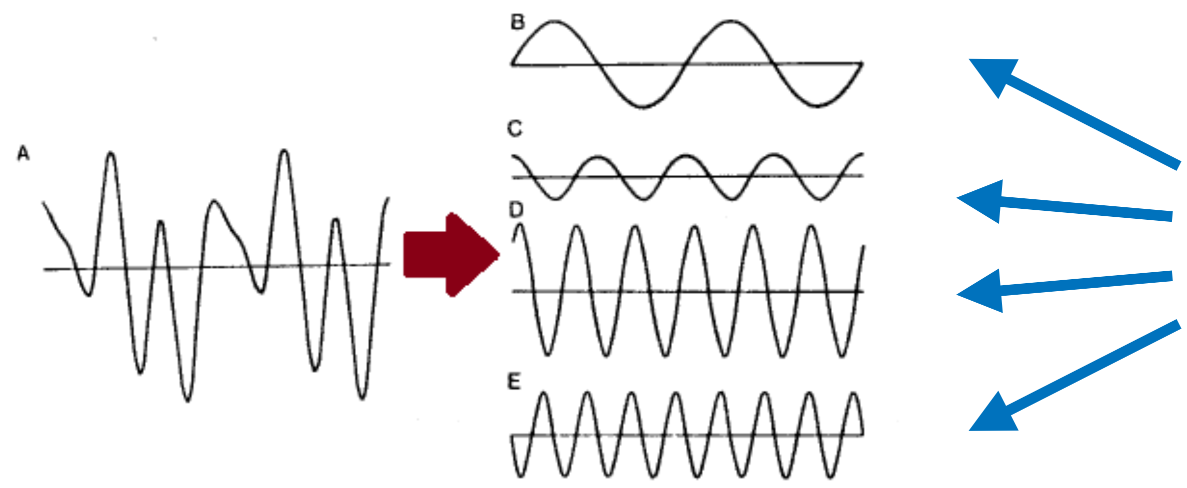

- "building blocks" of signal
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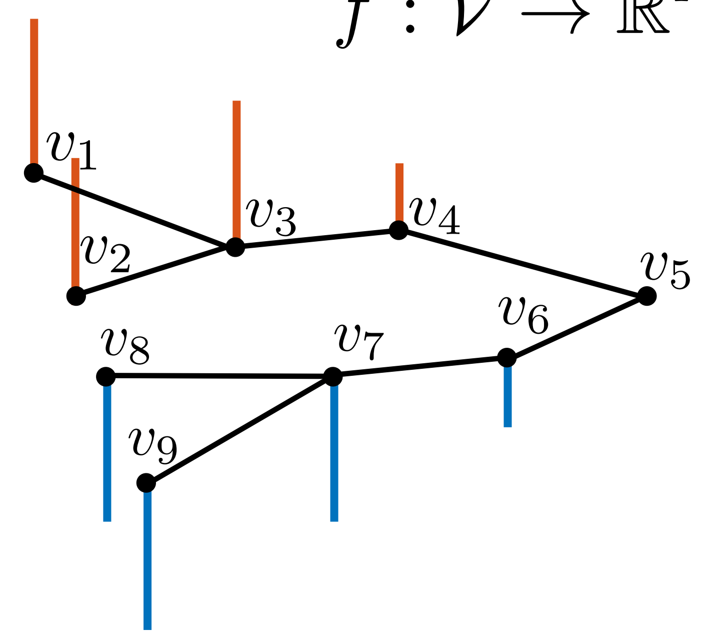
- What about a notion of frequency for graph signals?

$f : \mathcal{V} \to \mathbb{R}^N$



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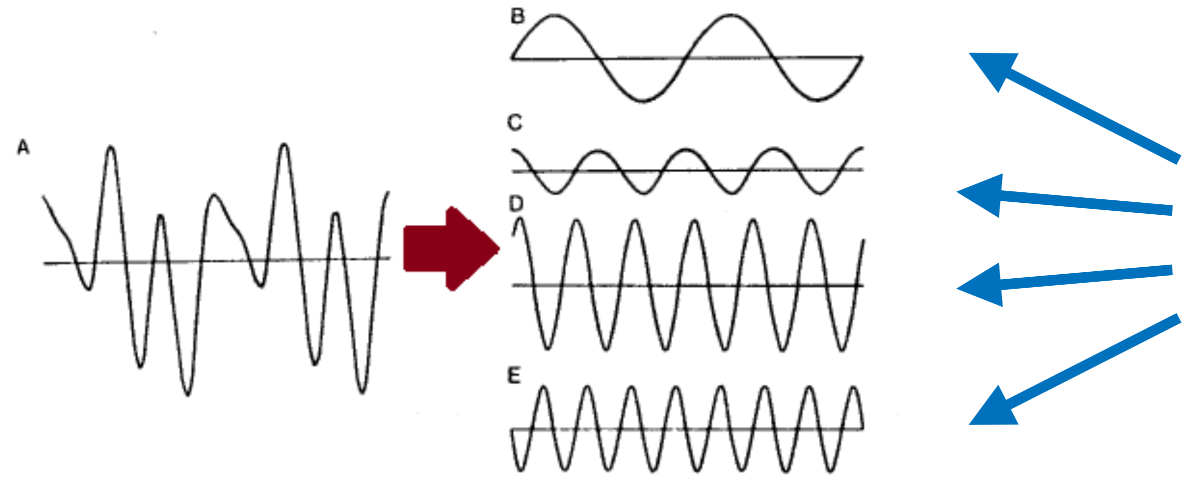

- "building blocks" of signal
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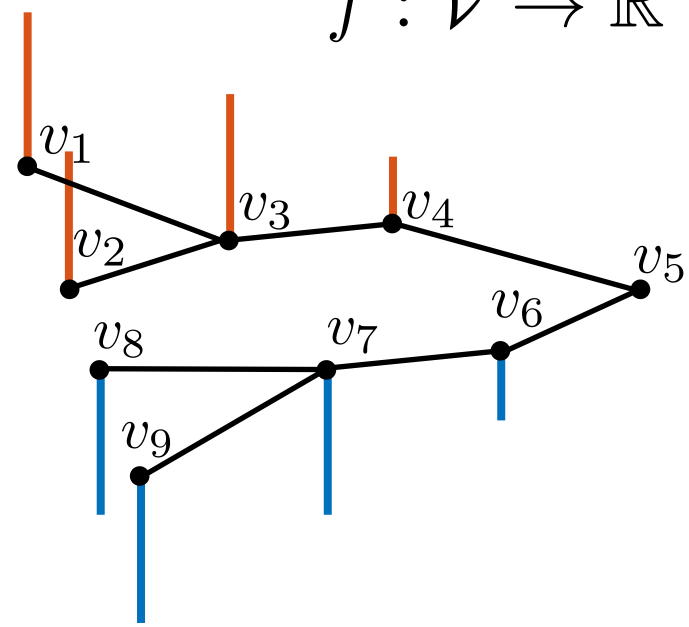
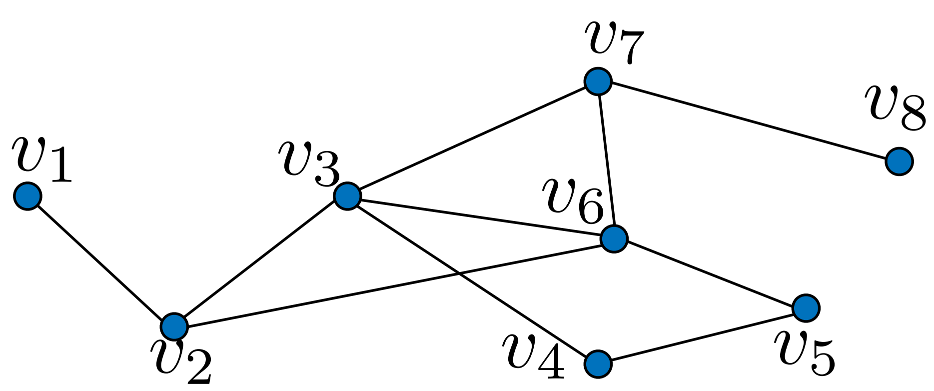
- What about a notion of frequency for graph signals?

  we need the graph Laplacian matrix

$f : \mathcal{V} \to \mathbb{R}^N$



Graph Laplacian



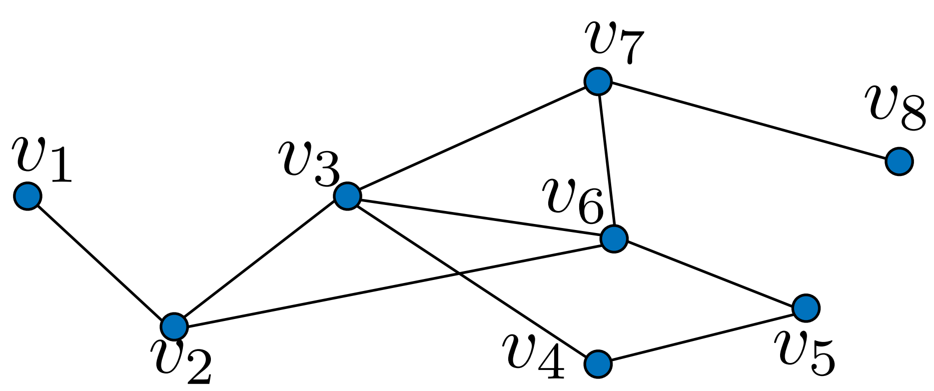
weighted and undirected graph:

$\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

W

Graph Laplacian



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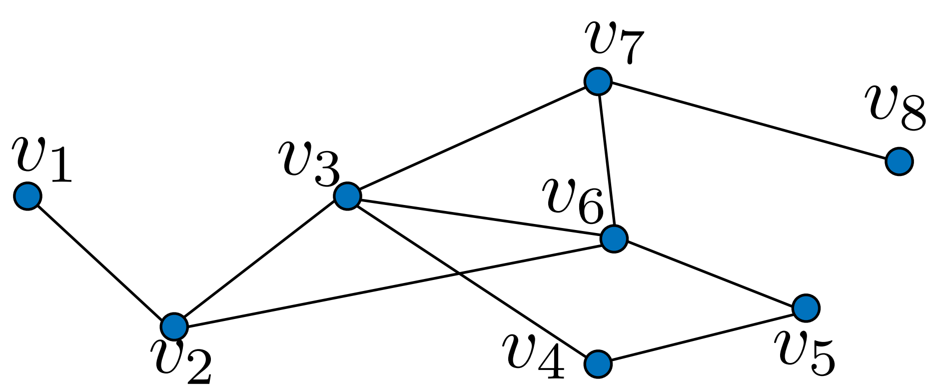
$$D = \mathrm{diag}(d(v_1), \cdots, d(v_N))$$

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad W = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Graph Laplacian



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$$L = D - W$$ **equivalent to G!**

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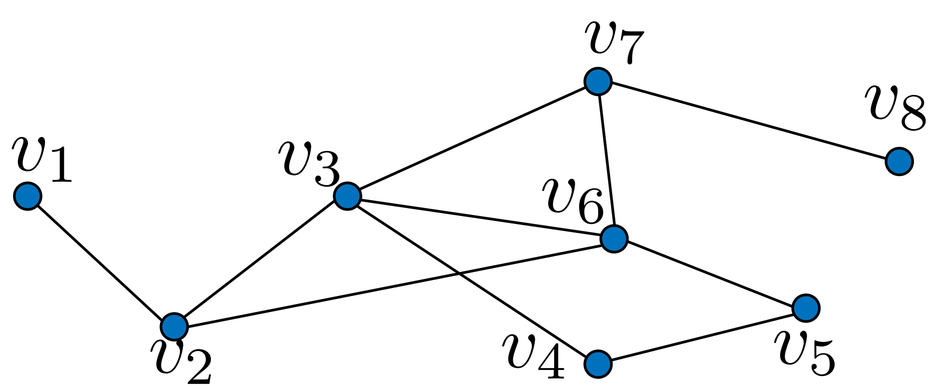
- symmetric
- off-diagonal entries non-positive
- rows sum up to zero

Graph Laplacian



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$$L_{\text{norm}} = D^{-\frac{1}{2}}(D - W)D^{-\frac{1}{2}}$$

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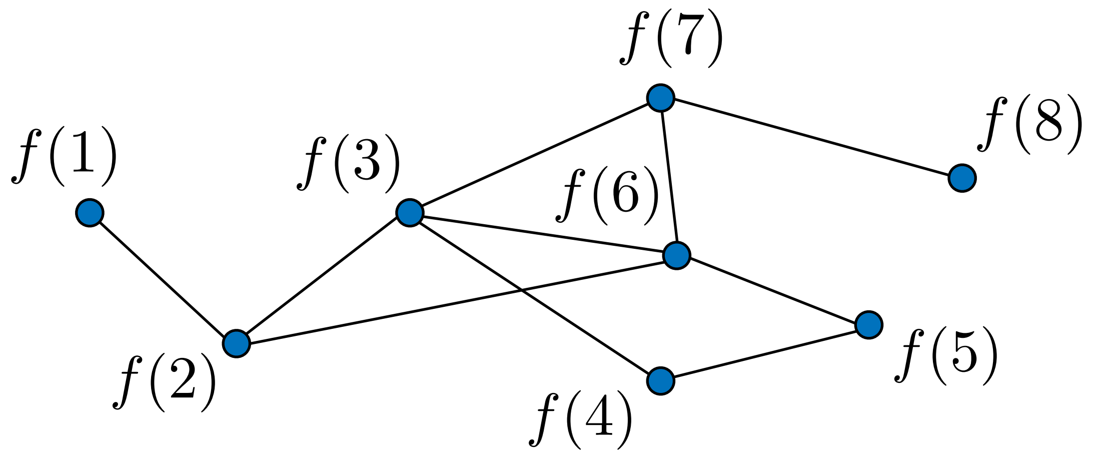
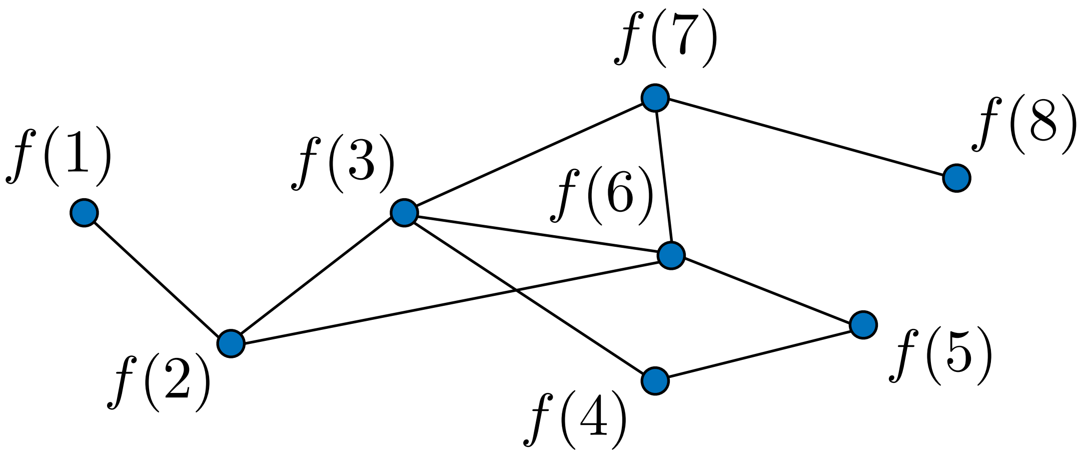


graph signal $f : \mathcal{V} \to \mathbb{R}^N$

Graph Laplacian



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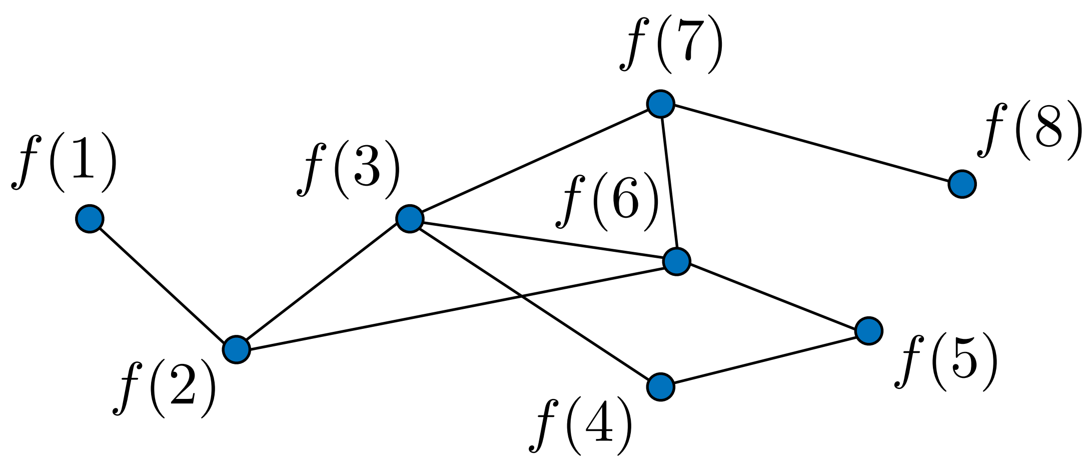
$$Lf(i) = \sum_{j=1}^{N} W_{ij}(f(i) - f(j))$$

Graph Laplacian



graph signal $f : \mathcal{V} \rightarrow \mathbb{R}^N$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \\ f(6) \\ f(7) \\ f(8) \end{pmatrix}$$

$$\begin{pmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \\ f(6) \\ f(7) \\ f(8) \end{pmatrix}^T \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \\ f(6) \\ f(7) \\ f(8) \end{pmatrix}$$

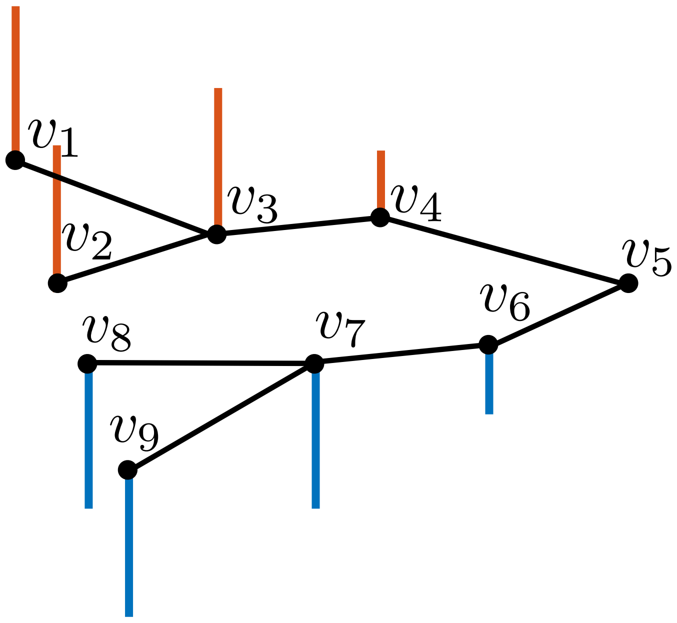
$$Lf(i) = \sum_{j=1}^{N} W_{ij}(f(i) - f(j))$$

$$f^T L f = \frac{1}{2} \sum_{i,j=1}^{N} W_{ij} \left(f(i) - f(j)\right)^2$$

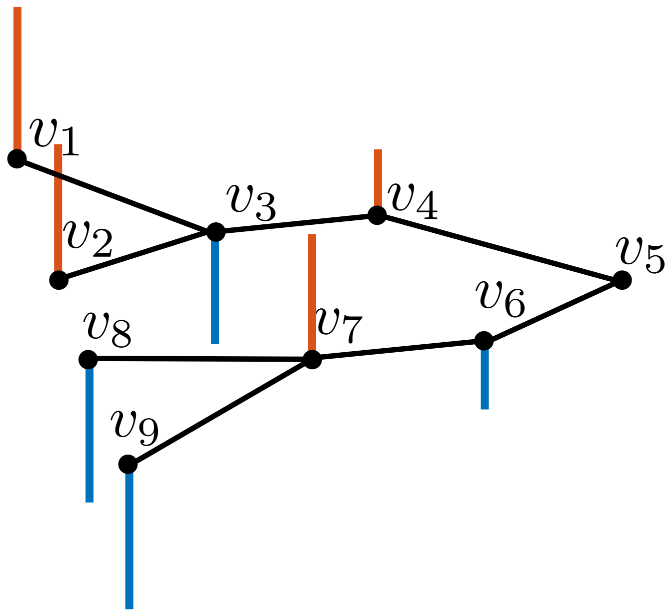
a measure of "smoothness"

Zhou and Schölkopf, "A regularization framework for learning from graph data," ICML Workshop, 2004.

Graph Laplacian



$$f^T L f = 1$$



$$f^T L f = 21$$

Graph Laplacian

- L has a complete set of orthonormal eigenvectors: $L = \chi \Lambda \chi^T$

$$L = \underbrace{\begin{bmatrix} | & & | \\ \chi_0 & \cdots & \chi_{N\text{-}1} \\ | & & | \end{bmatrix}}_{\chi} \underbrace{\begin{bmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_{N-1} \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} — & \chi_0^T & — \\ & \cdots & \\ — & \chi_{N\text{-}1}^T & — \end{bmatrix}}_{\chi^T}$$

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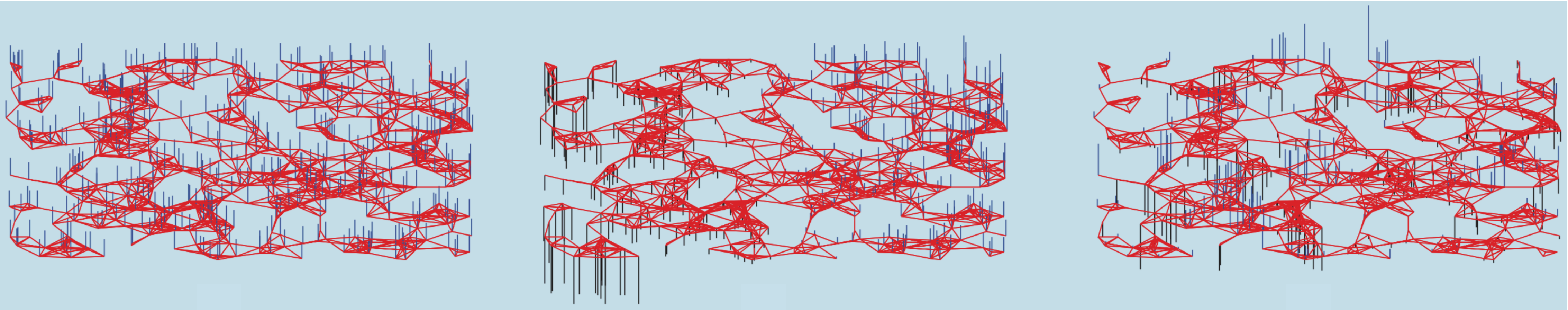
- Eigenvalues are usually sorted increasingly: $0 = \lambda_0 < \lambda_1 \leq \ldots \leq \lambda_{N-1}$

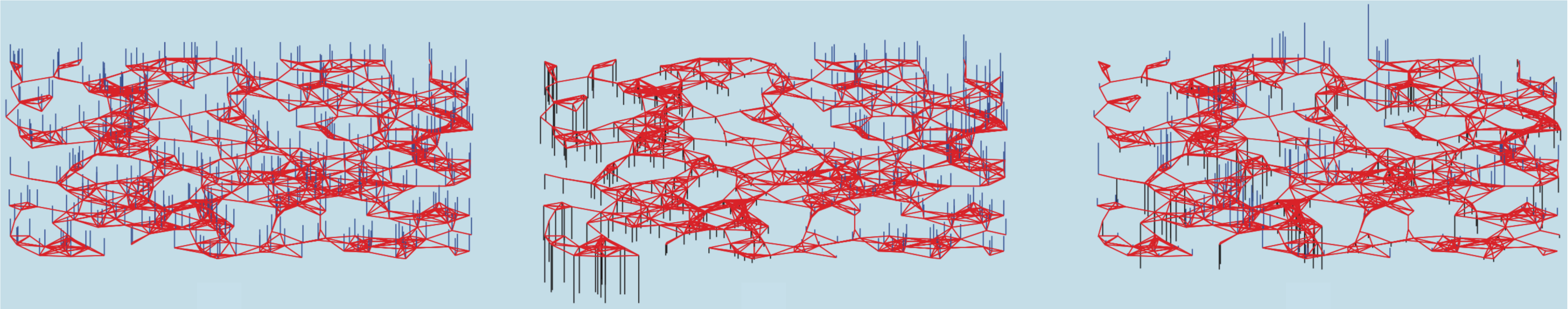
Graph Fourier transform



χ_0 χ_1 χ_{50}

Shuman et al., "The emerging field of signal processing on graphs," IEEE SPM, 2013.

Graph Fourier transform



χ_0 χ_1 χ_{50}

low frequency → high frequency

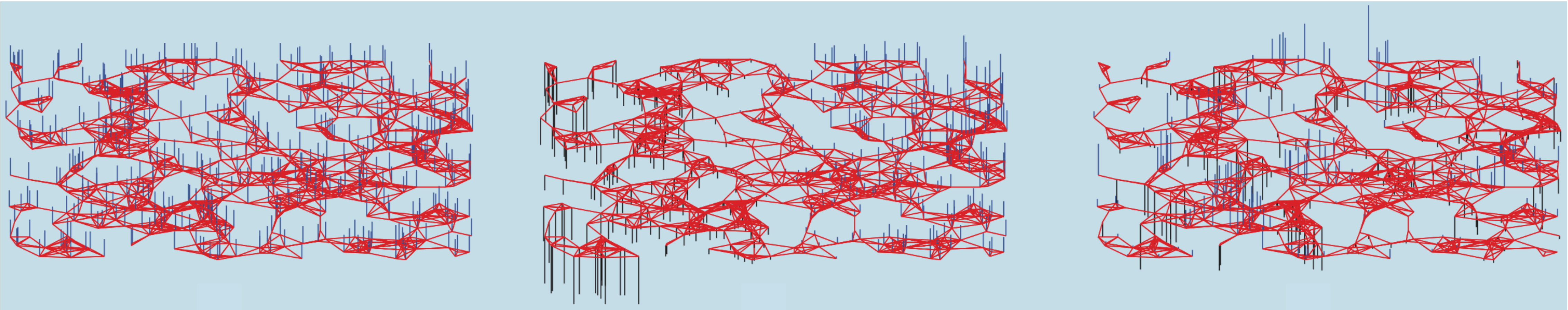
$L = \chi \Lambda \chi^T$

$\chi_0^T L \chi_0 = \lambda_0 = 0$ $\chi_{50}^T L \chi_{50} = \lambda_{50}$

- Eigenvectors associated with smaller eigenvalues have values that vary less rapidly along the edges

Shuman et al., "The emerging field of signal processing on graphs," IEEE SPM, 2013.

Graph Fourier transform



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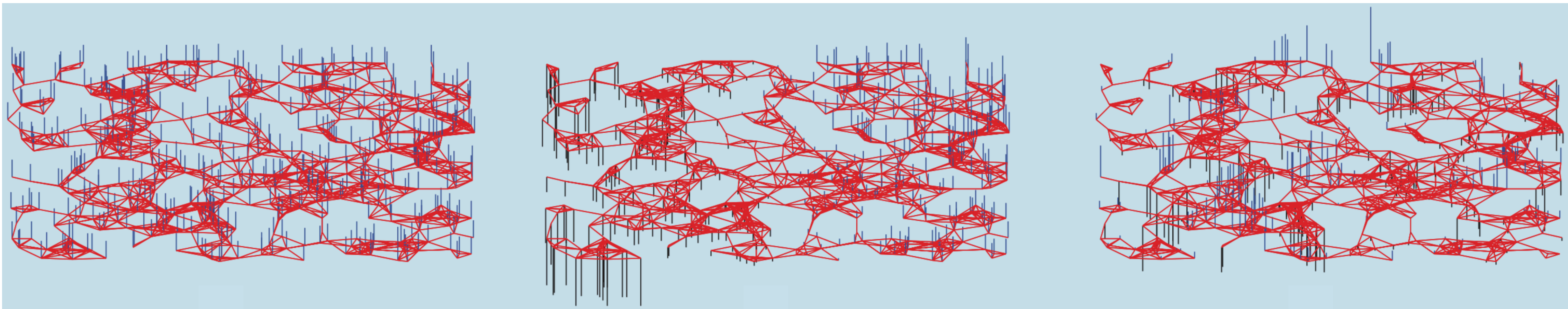
graph Fourier transform:
[Hammond11]

$$\hat{f}(\ell) = \langle \chi_\ell, f \rangle : \quad \begin{bmatrix} | & & | \\ \chi_0 & \cdots & \chi_{N\text{-}1} \\ | & & | \end{bmatrix}^T \begin{matrix} | \\ f \\ | \end{matrix}$$

Shuman et al., "The emerging field of signal processing on graphs," IEEE SPM, 2013.

Graph Fourier transform



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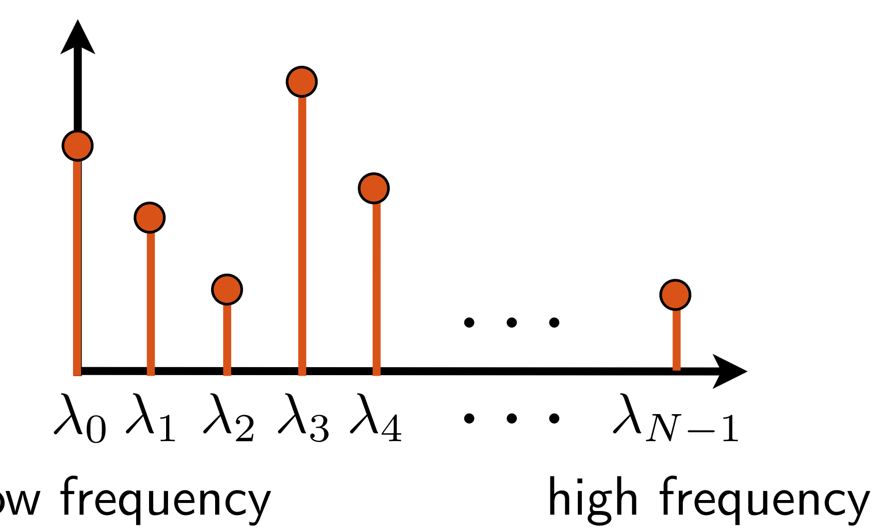
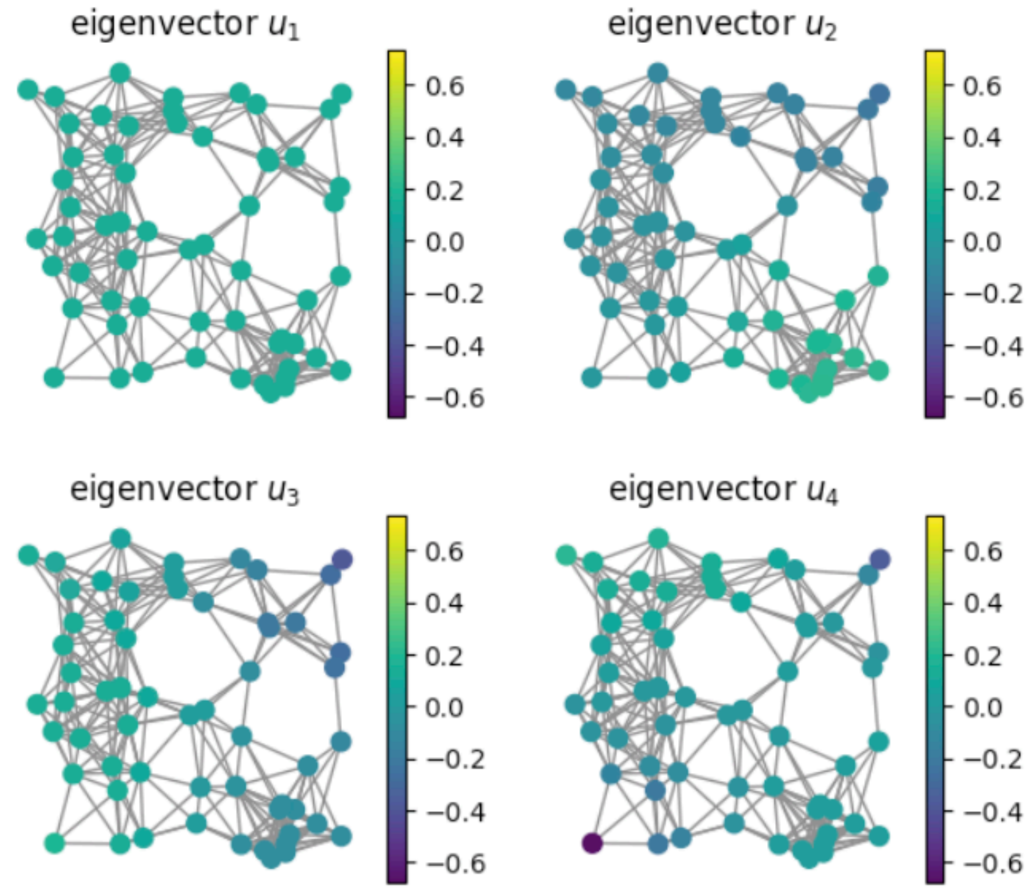


Shuman et al., "The emerging field of signal processing on graphs," IEEE SPM, 2013.

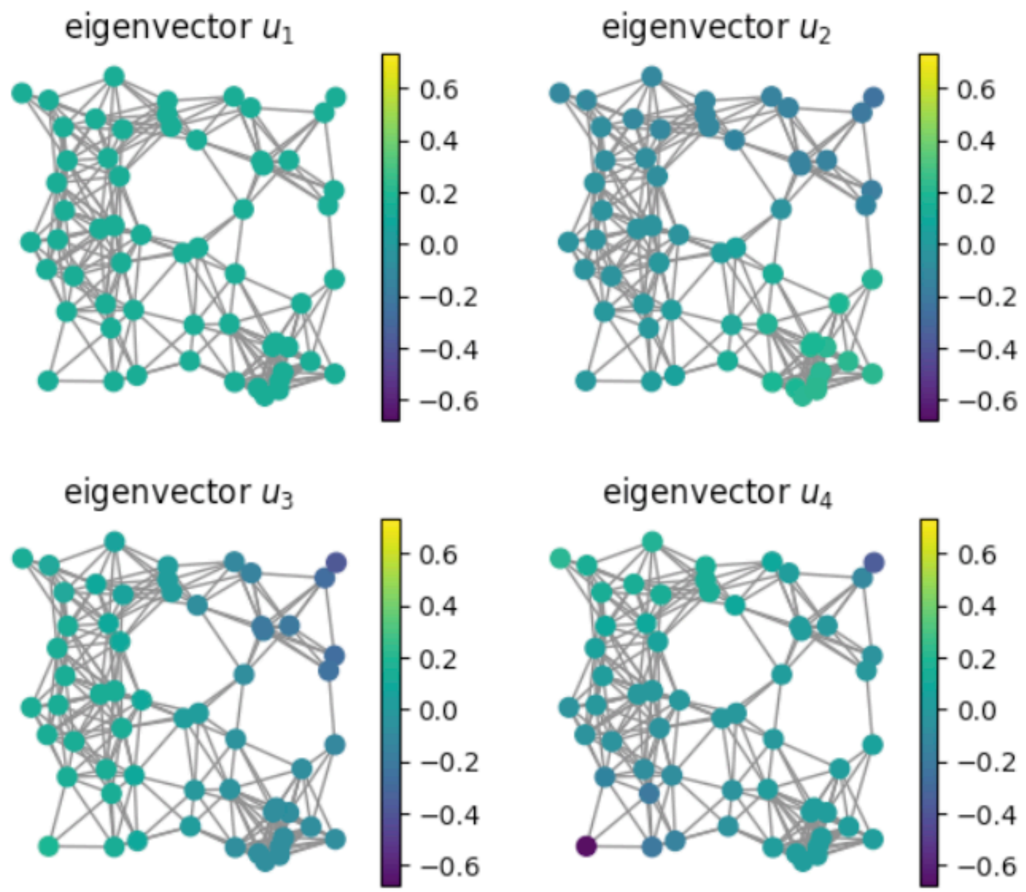
Graph Fourier transform

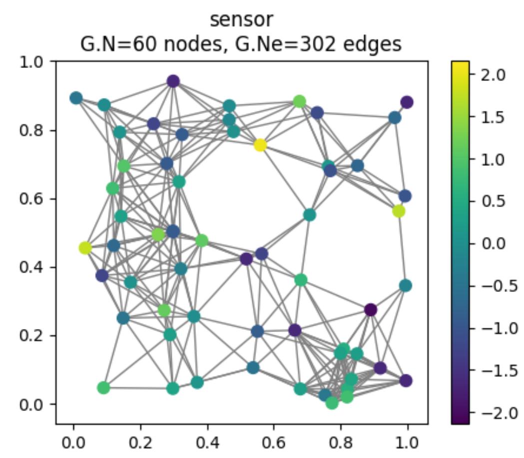
- Example on a simple graph

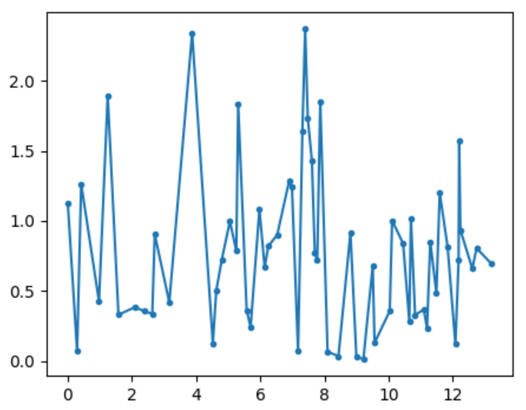


Graph Fourier transform

- Example on a simple graph







Graph Fourier transform

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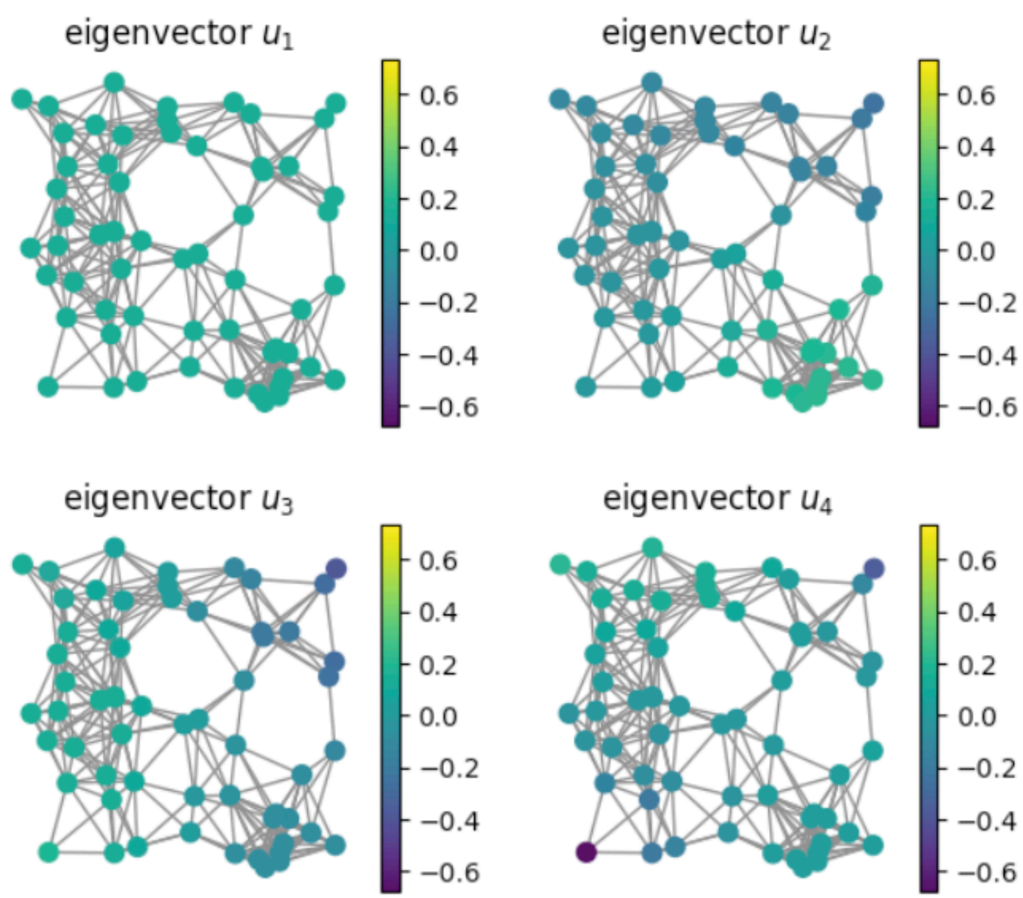

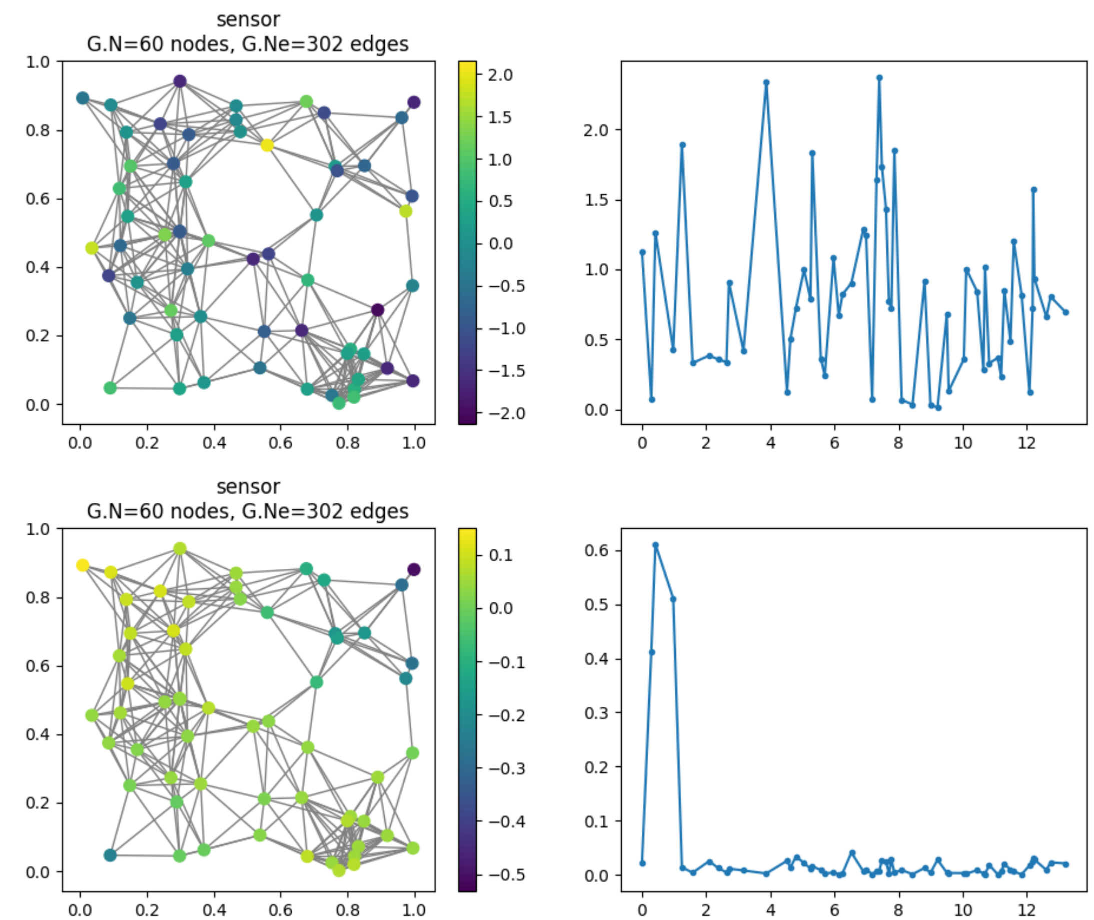

Outline

- Graph signal processing (GSP): Basic concepts
- Filtering of graph signals: Basic tool of GSP
- Connection with literature
- Research topics inspired by GSP
- Applications

Filtering of graph signals

GFT: $\hat{f}(\ell) = \langle \chi_\ell, f \rangle = \sum_{i=1}^{N} \chi_\ell^*(i) f(i) \qquad f(i) = \sum_{\ell=0}^{N-1} \hat{f}(\ell) \chi_\ell(i)$

Filtering of graph signals

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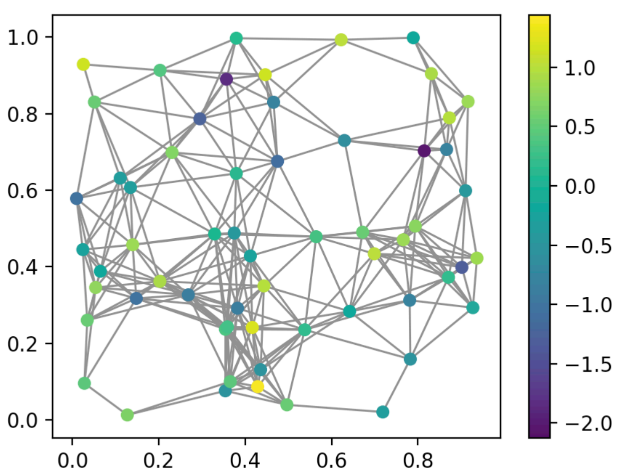
f



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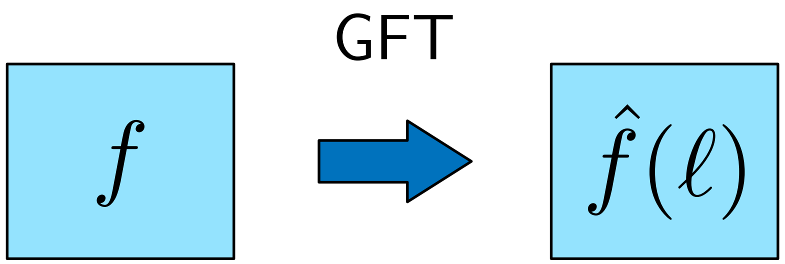

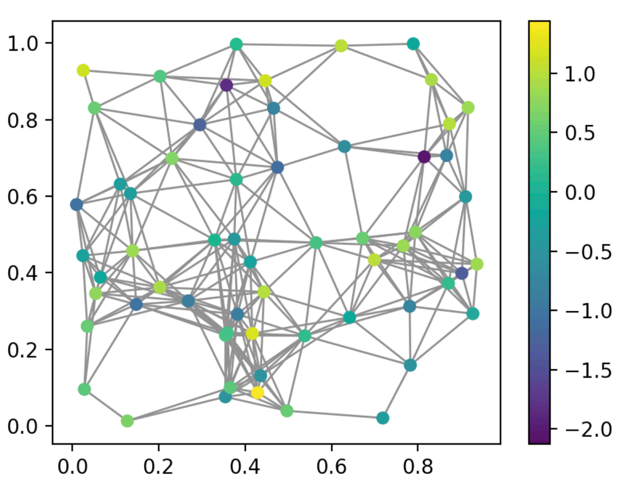

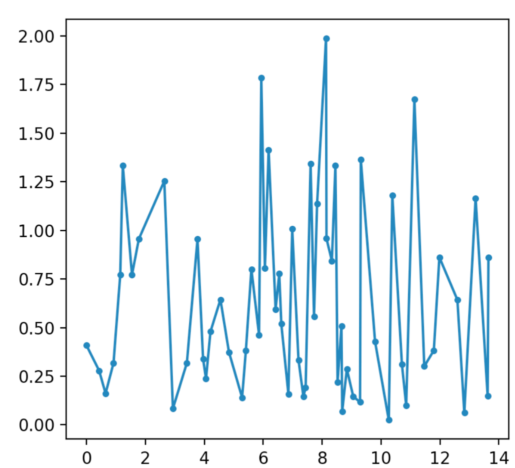

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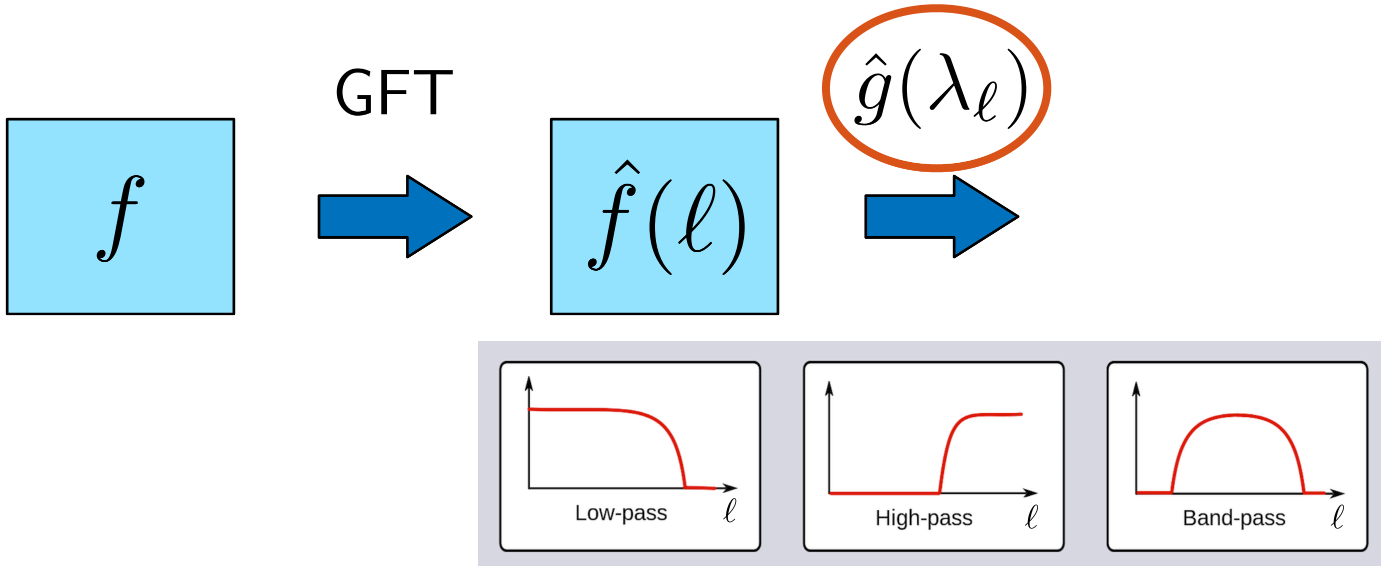

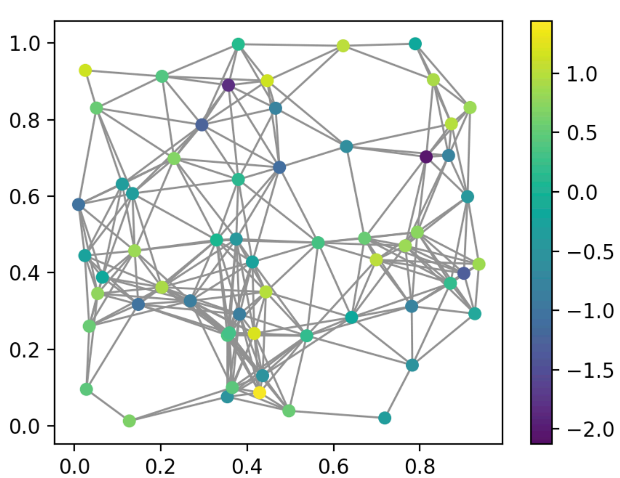

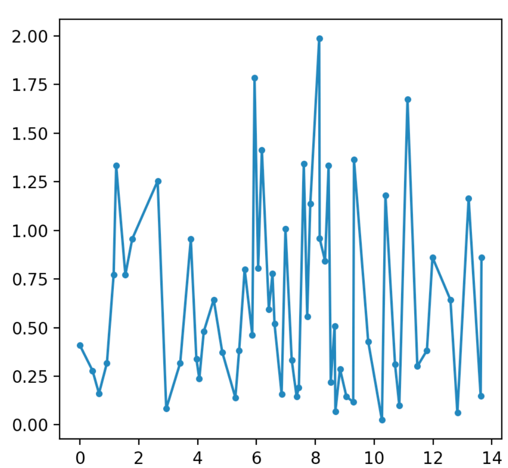

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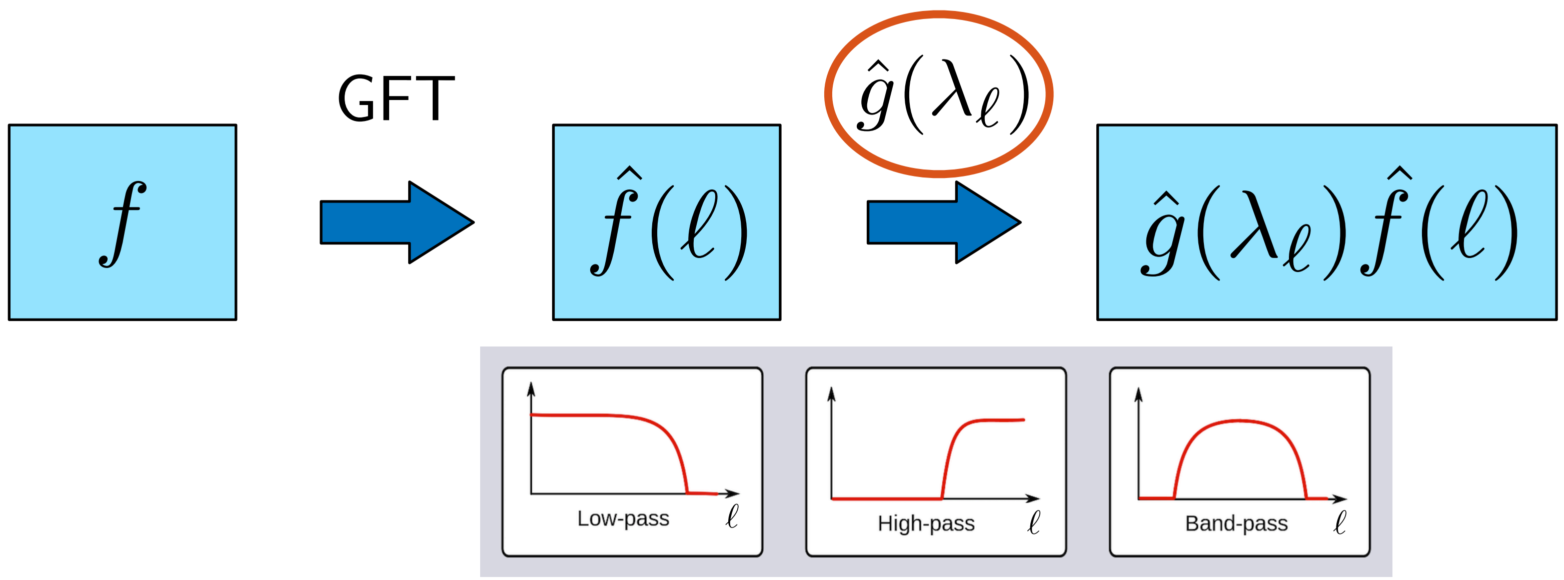

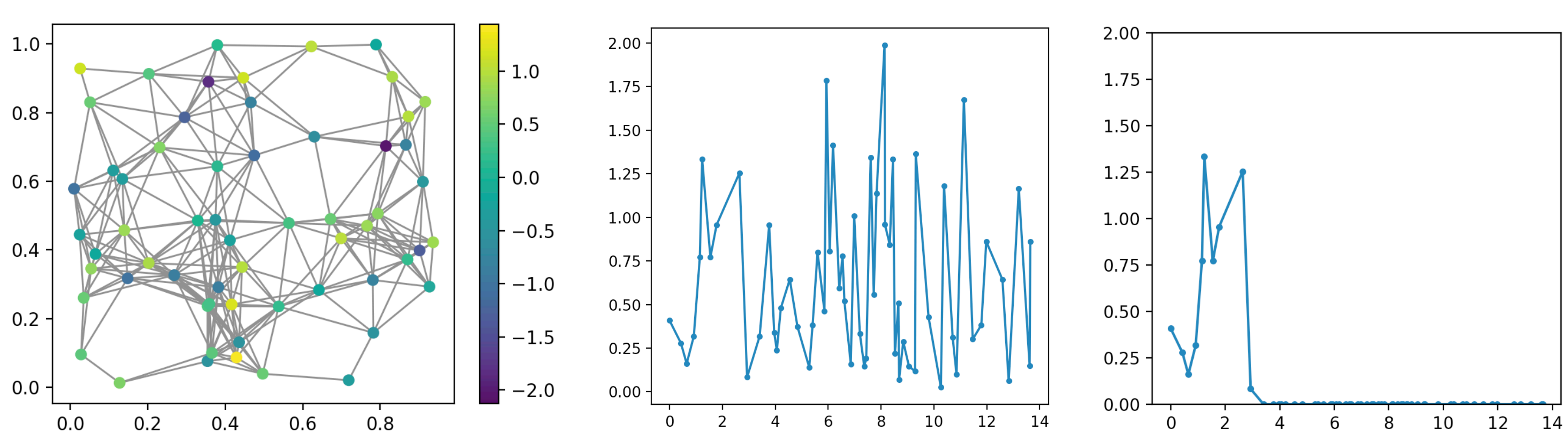

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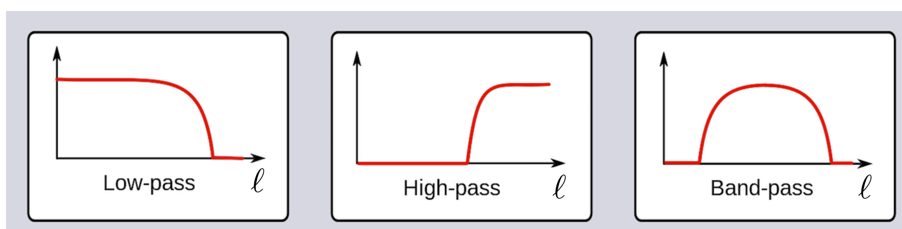
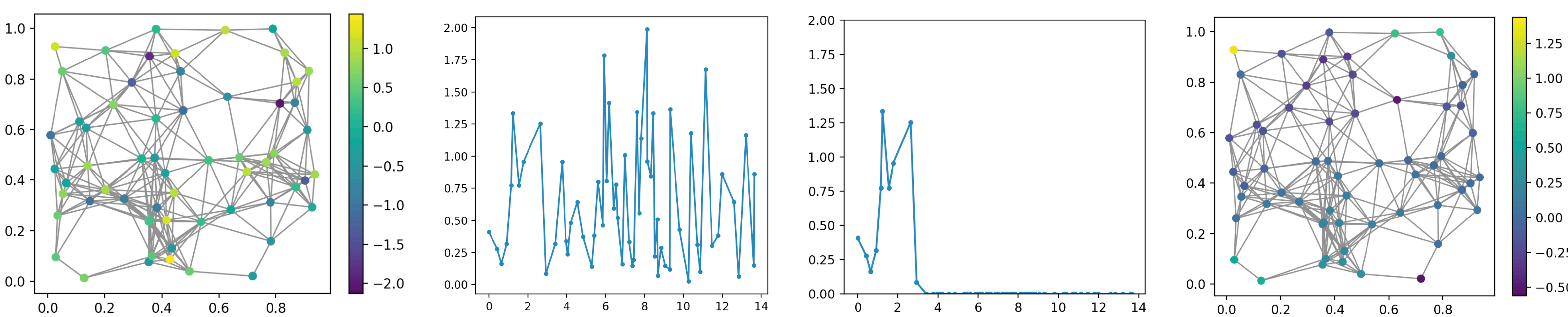
f → GFT → $\hat{f}(\ell)$ → $\hat{g}(\lambda_\ell)$ → $\hat{g}(\lambda_\ell)\hat{f}(\ell)$ → IGFT → $f(i) = \sum_{\ell=0}^{N-1} \hat{g}(\lambda_\ell)\hat{f}(\ell)\chi_\ell(i)$





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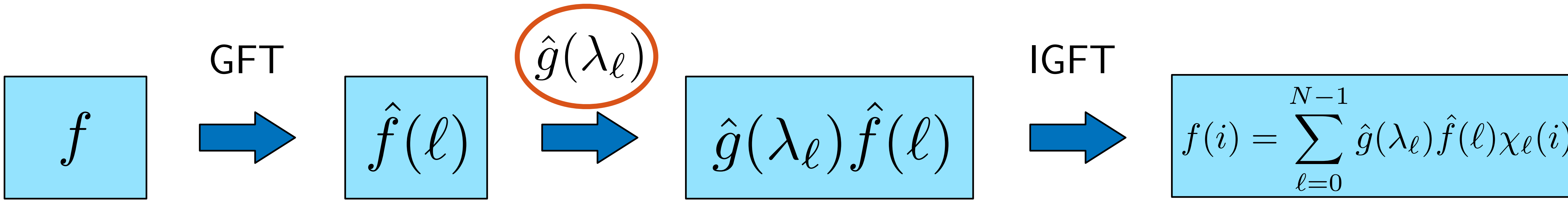

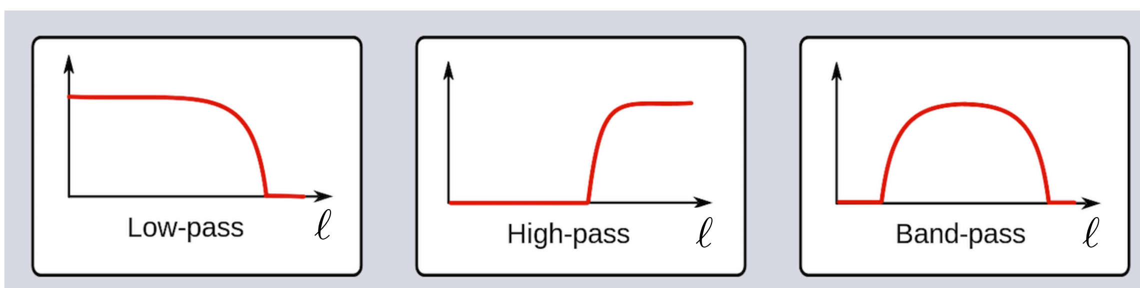

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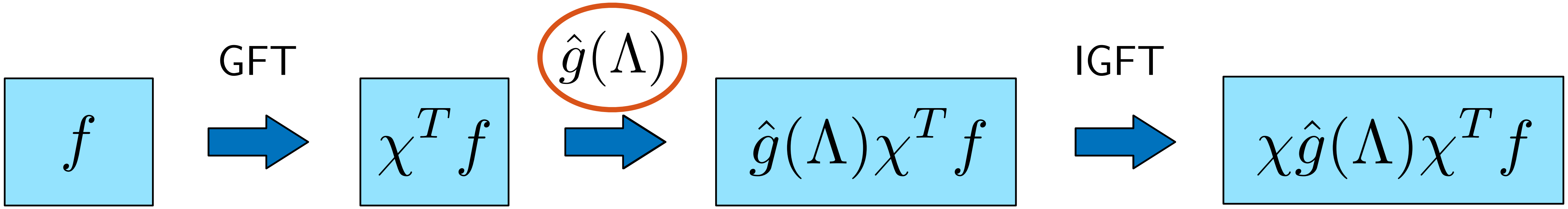

$$\hat{g}(\Lambda) = \begin{bmatrix} \hat{g}(\lambda_0) & & 0 \\ & \ddots & \\ 0 & & \hat{g}(\lambda_{N-1}) \end{bmatrix}$$

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f → GFT → $\chi^T f$ → $\hat{g}(\Lambda)$ → $\hat{g}(\Lambda)\chi^T f$ → IGFT → $\chi \hat{g}(\Lambda) \chi^T f$

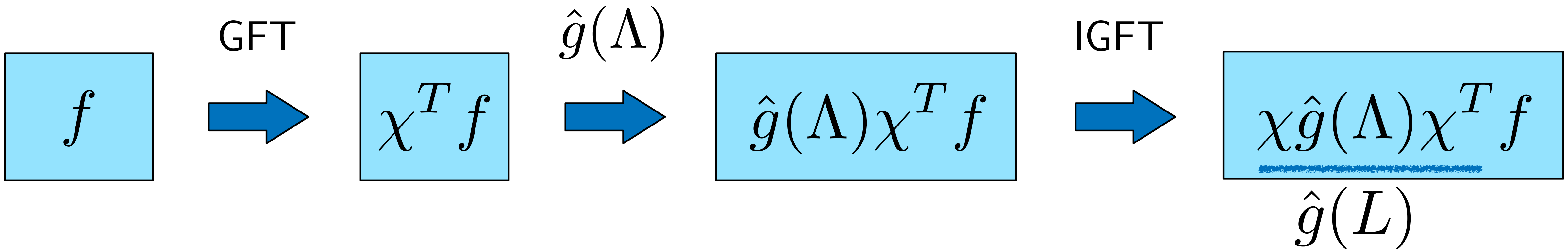
$\hat{g}(L)$**: functions of L!**

$$\hat{g}(\Lambda) = \begin{bmatrix} \hat{g}(\lambda_0) & & 0 \\ & \ddots & \\ 0 & & \hat{g}(\lambda_{N-1}) \end{bmatrix}$$

A practical example

f $\xrightarrow{\text{GFT}}$ $\chi^T f$ $\xrightarrow{\hat{g}(\Lambda)}$ $\hat{g}(\Lambda)\chi^T f$ $\xrightarrow{\text{IGFT}}$ $\underbrace{\chi \hat{g}(\Lambda)\chi^T}_{\hat{g}(L)} f$

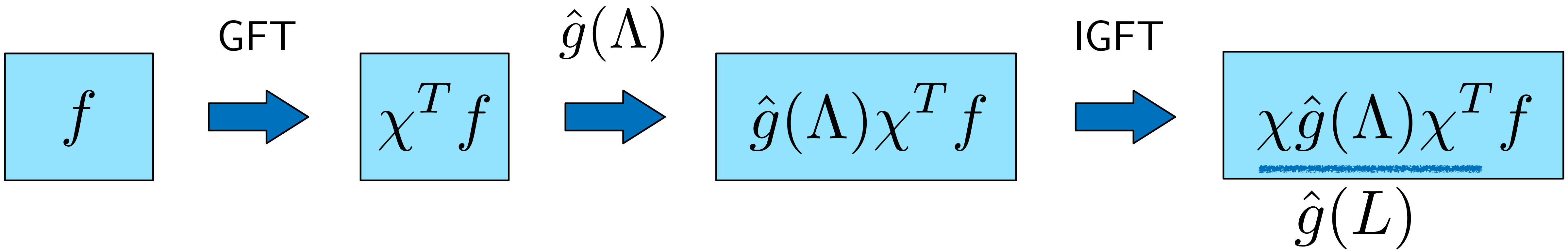
A practical example



problem: we observe a noisy graph signal $f = y_0 + \eta$ and wish to recover y_0

$$y^* = \arg\min_y \{||y - f||_2^2 + \gamma y^T L y\}$$

A practical example



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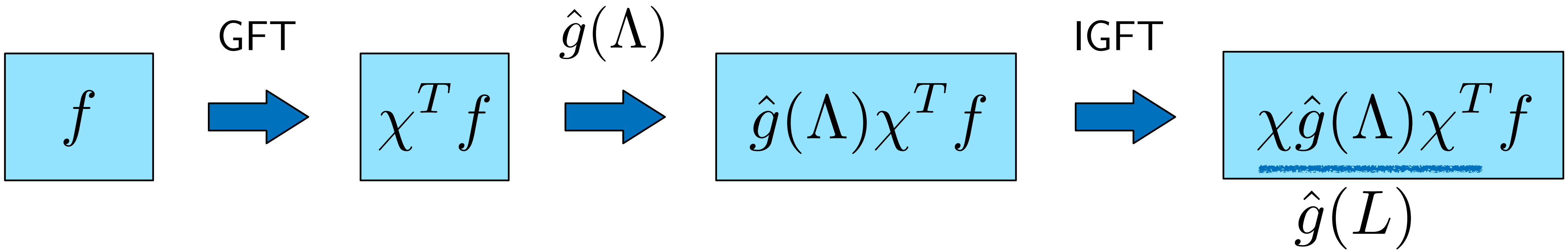
data fitting term

"smoothness" assumption

A practical example



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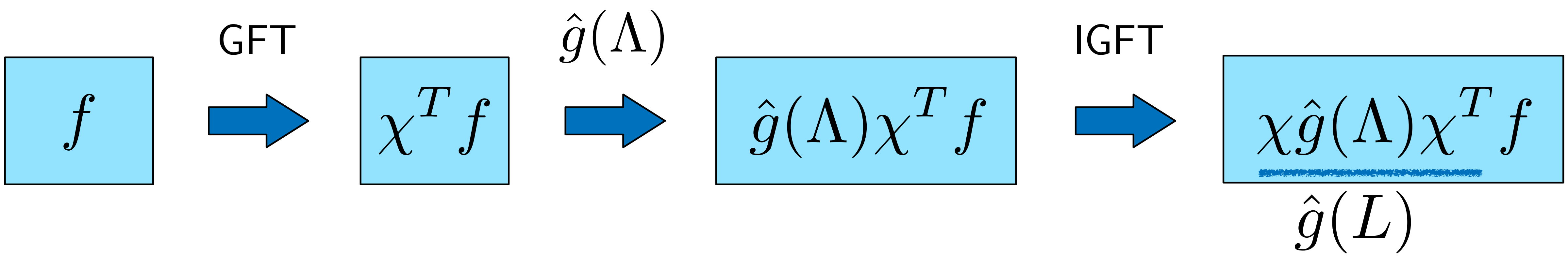
data fitting term: $||y - f||_2^2$

"smoothness" assumption: $\gamma y^T L y$

$$y^* = \underbrace{(I + \gamma L)^{-1}}_{\hat{g}(L)} f$$

A practical example



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data fitting term

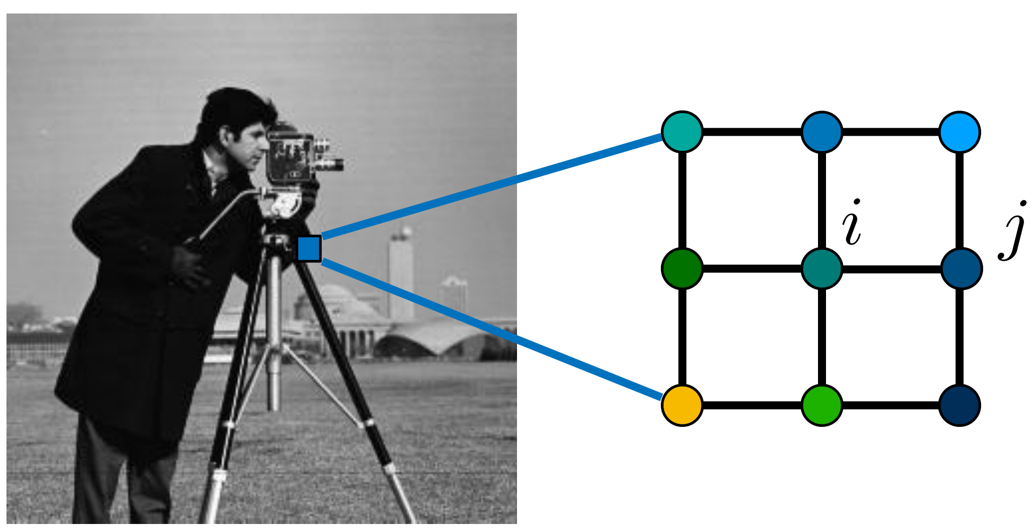
"smoothness" assumption

$$y^* = \underbrace{(I + \gamma L)^{-1}}_{\hat{g}(L)} f = \chi(1 + \gamma\Lambda)^{-1}\chi^T f$$

remove noise by low-pass filtering in graph spectral domain!

A practical example

- noisy image as observed noisy graph signal
- regular grid graph (weights inversely proportional to pixel value difference)



$$w_{ij} = \frac{1}{|f(i) - f(i)|}$$

Shuman et al., "The emerging field of signal processing on graphs," IEEE SPM, 2013.

A practical example

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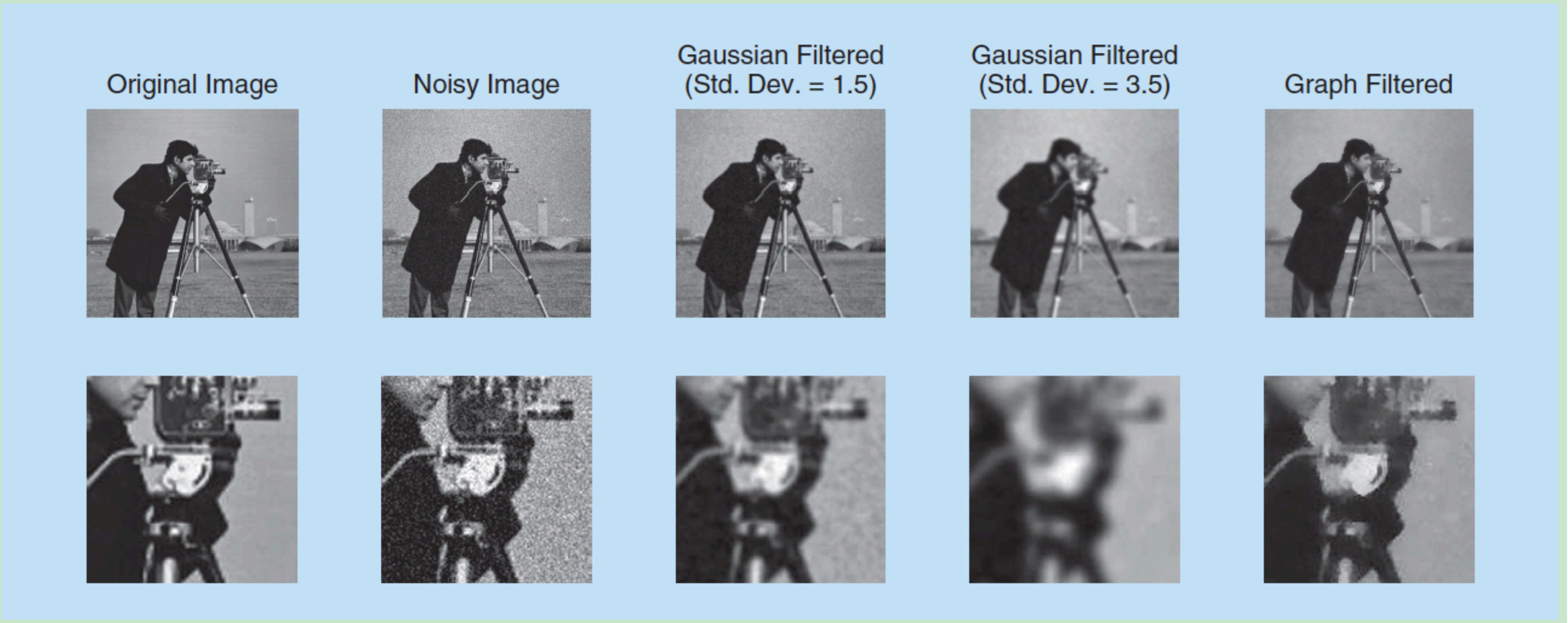
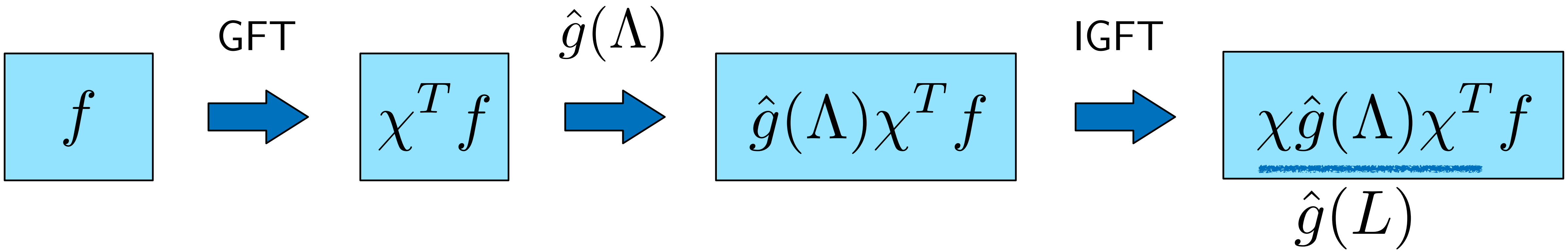


[FIGS2] **Image denoising via classical Gaussian filtering and graph spectral filtering.**

Shuman et al., "The emerging field of signal processing on graphs," IEEE SPM, 2013.

More filtering operations



low-pass filters: $\hat{g}(L) = (I + \gamma L)^{-1} = \chi(I + \gamma\Lambda)^{-1}\chi^T$

window kernel: windowed graph Fourier transform

shifted and dilated band-pass filters: spectral graph wavelets $\hat{g}(sL)$

adapted kernels: learn values of $\hat{g}(L)$ directly from data

parametric polynomials: $\hat{g}_s(L) = \sum_{k=0}^{K} \alpha_{sk} L^k = \chi(\sum_{k=0}^{K} \alpha_{sk} \Lambda^k)\chi^T$

Shuman, "Dictionary design for graph signal processing," GSP Workshop, 2016.

Outline

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GSP and the literature

there is a rich literature about data analysis and learning on graphs

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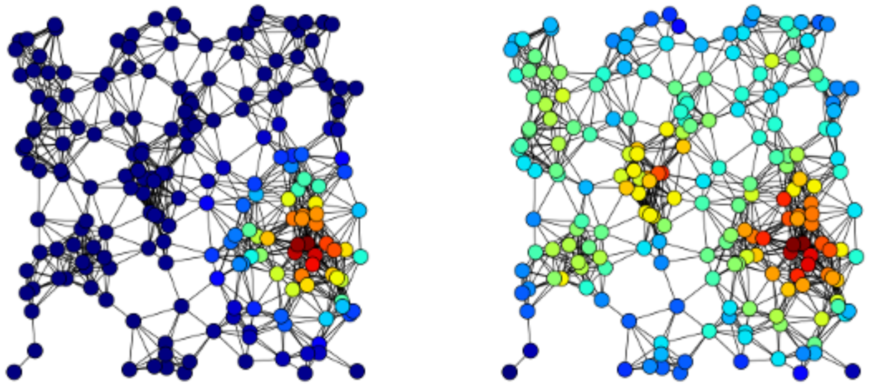
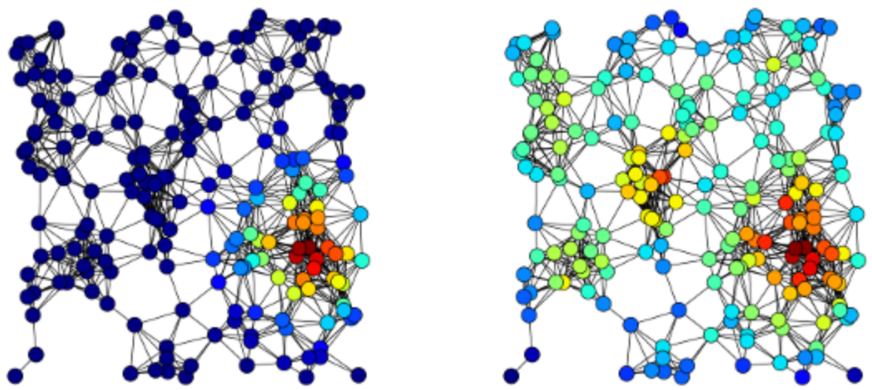


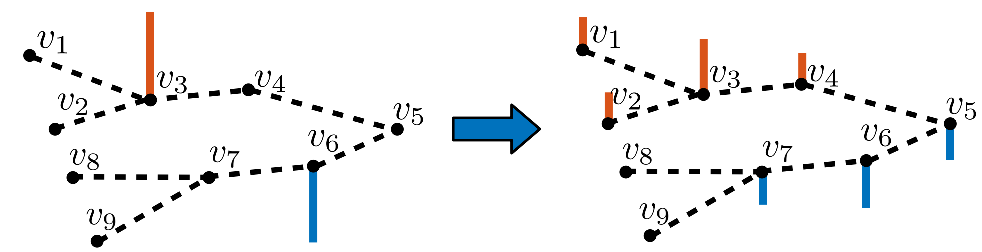
network science

GSP and the literature

there is a rich literature about data analysis and learning on graphs



network science



diffusion on graphs

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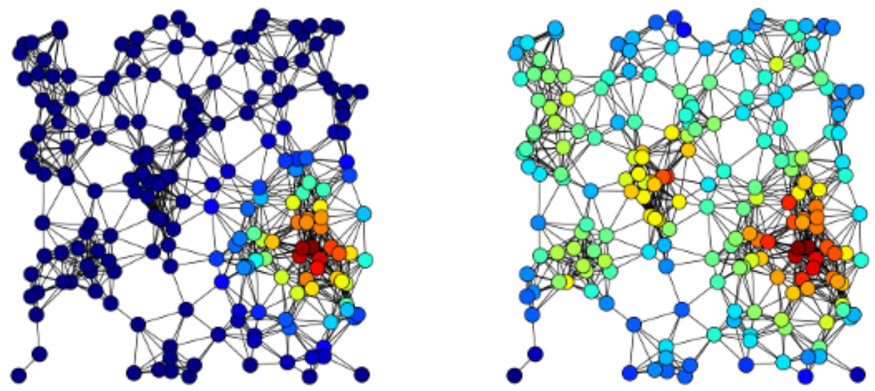
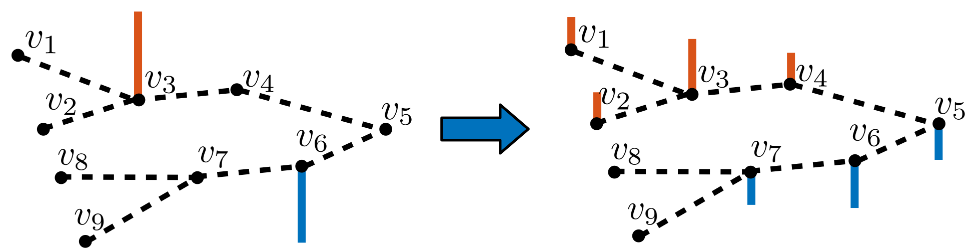


network science



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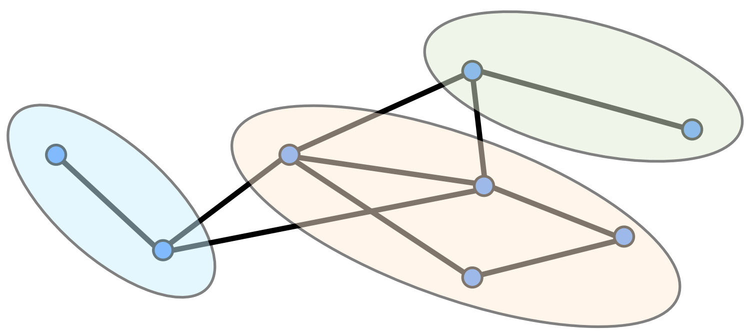

unsupervised learning (dimensionality reduction, clustering)

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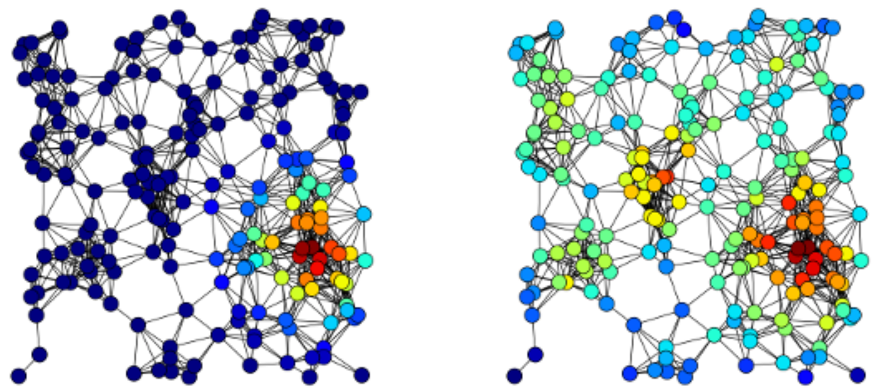

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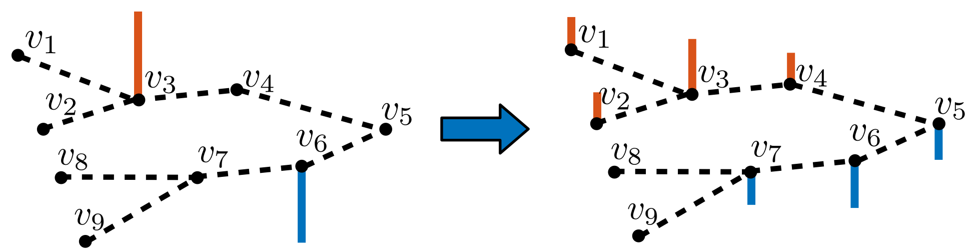

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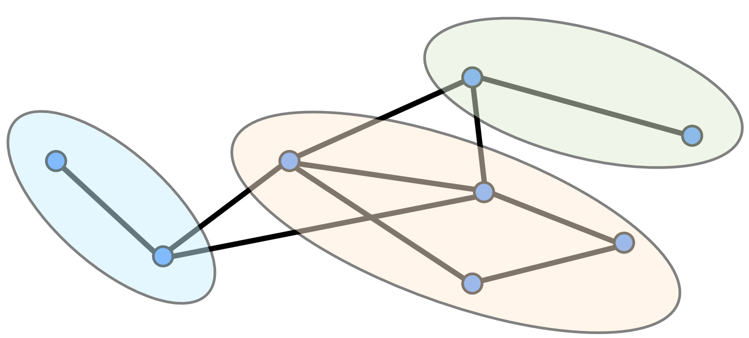
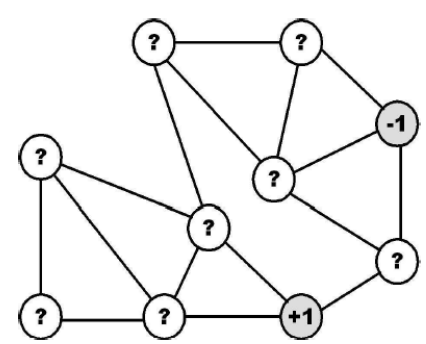


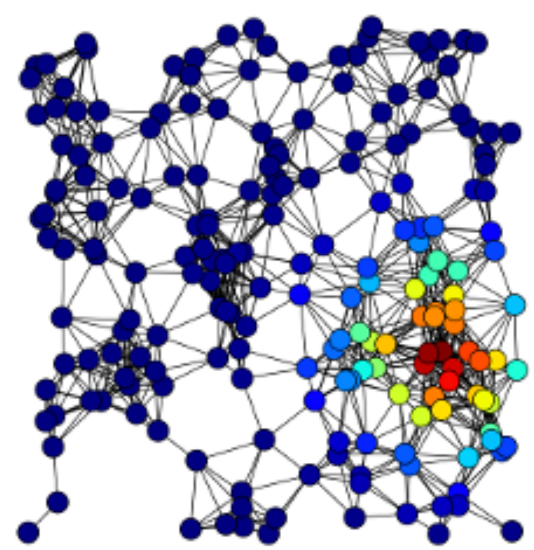
unsupervised learning (dimensionality reduction, clustering)



semi-supervised learning

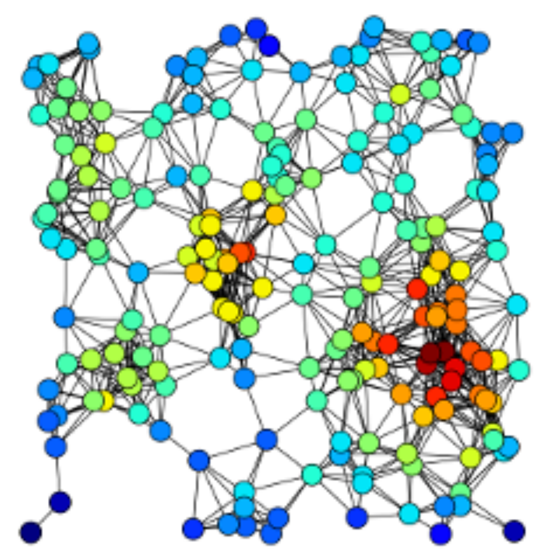
Network centrality

eigenvector centrality



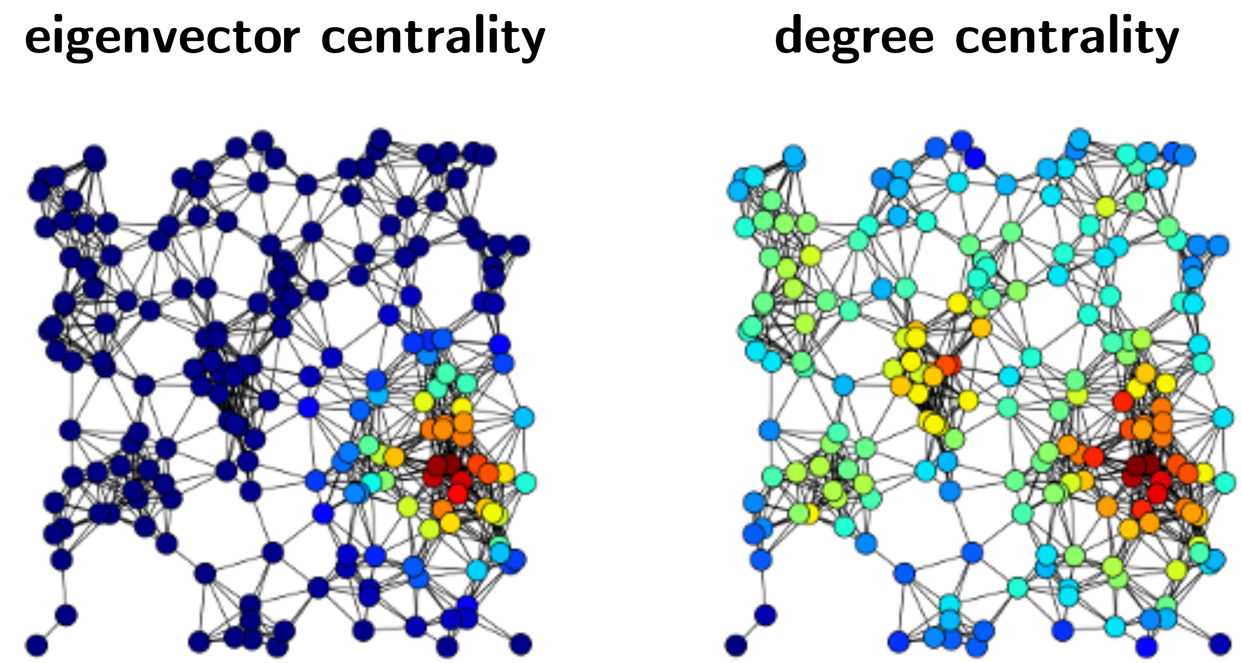
$$Wx = \lambda_{\max} x$$

degree centrality



$$d = [d(v_1), \cdots, d(v_N)]$$

Network centrality

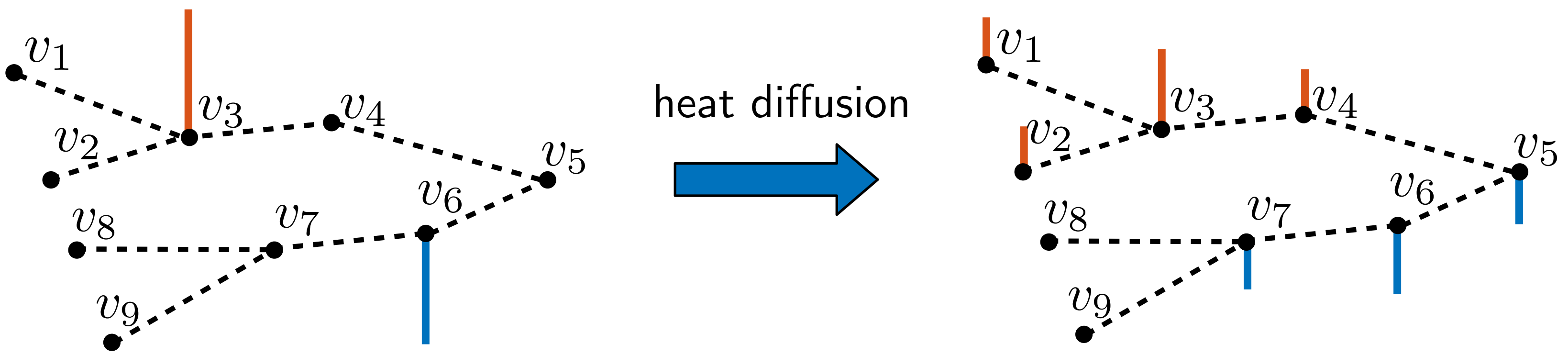


$$Wx = \lambda_{\max} x \qquad d = [d(v_1), \cdots, d(v_N)]$$

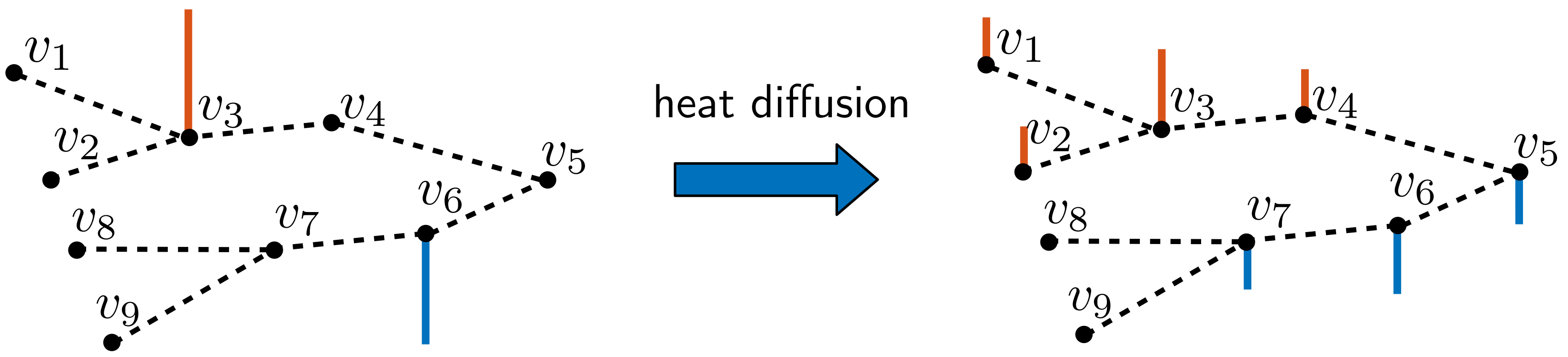
- Google's PageRank is a variant of eigenvector centrality
- eigenvectors of W can also be used to provide a frequency interpretation for graph signals

PageRank: http://www.ams.org/publicoutreach/feature-column/fcarc-pagerank
Sandryhaila and Moura, "Discrete signal processing on graphs," IEEE TSP, 2013.

Diffusion on graphs



Diffusion on graphs

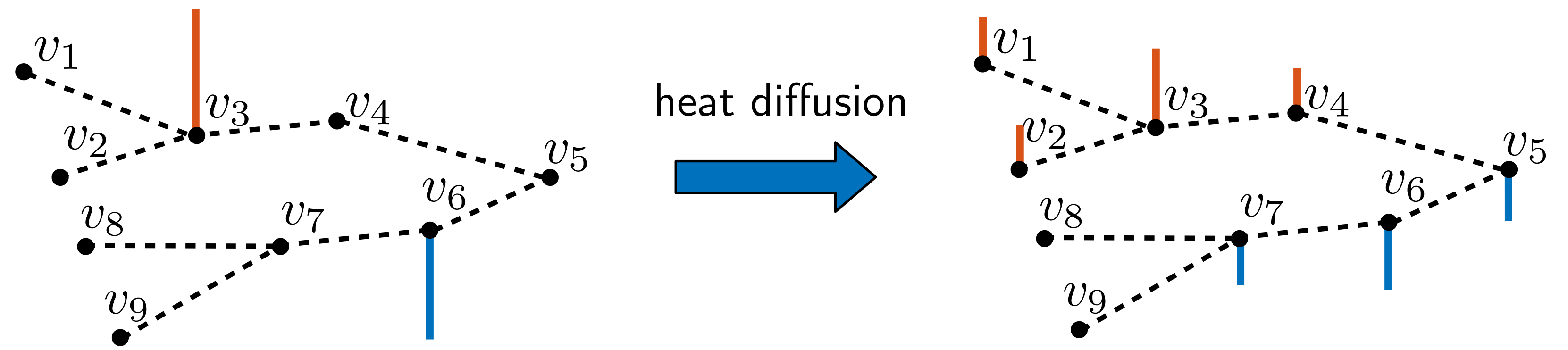


$$\frac{\partial x}{\partial \tau} - Lx = 0$$

$$x(v, 0) = x_0(v)$$

$$\Longrightarrow \quad x(v, \tau) = e^{-\tau L} x_0(v)$$

Diffusion on graphs



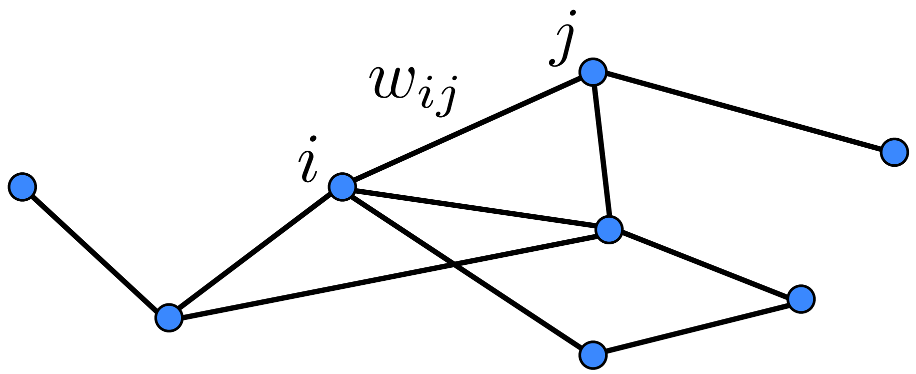
$$\frac{\partial x}{\partial \tau} - Lx = 0$$
$$x(v, 0) = x_0(v)$$

$$\Rightarrow \quad x(v, \tau) = e^{-\tau L} x_0(v)$$

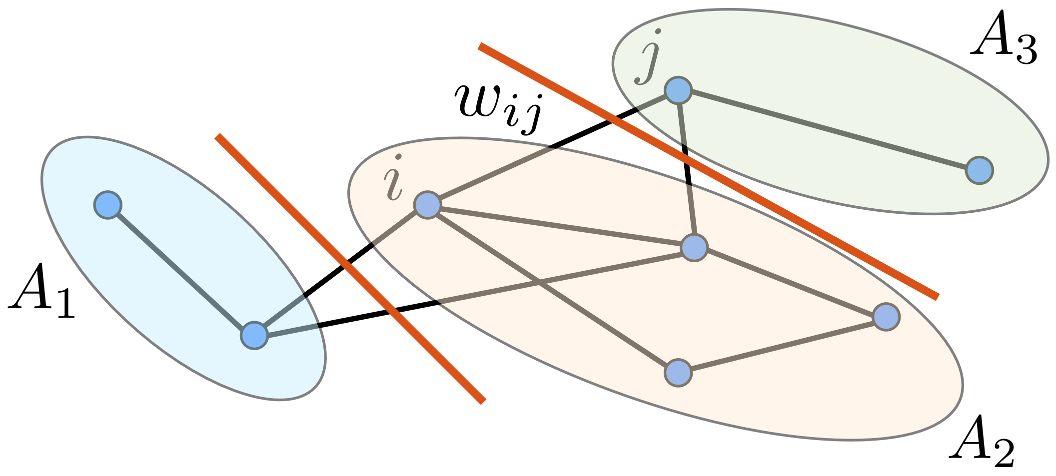
- heat diffusion on graphs is a typical physical process on graphs
- other possibilities exist (e.g., random walk on graphs)
- many have an interpretation of filtering on graphs

Smola and Kondor, "Kernels and regularization on graphs," COLT, 2003.

Graph clustering (community detection)



Graph clustering (community detection)



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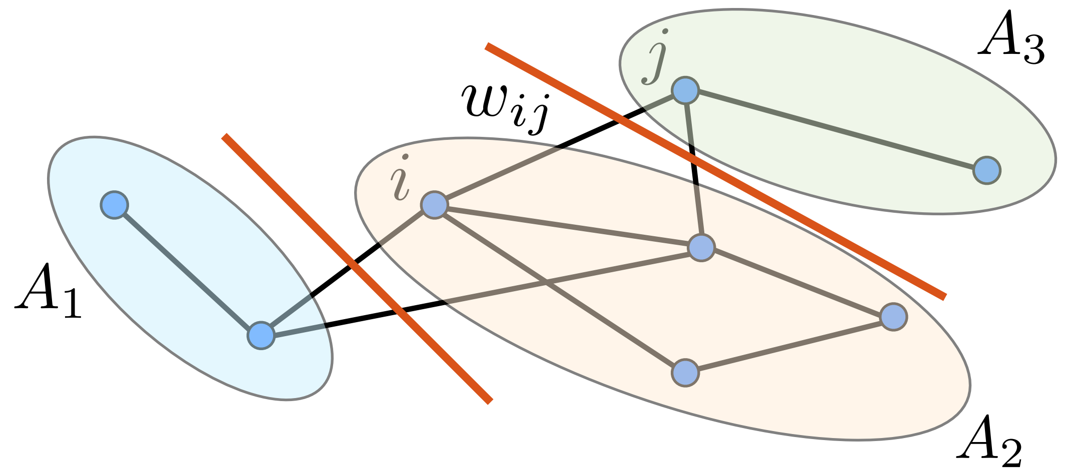
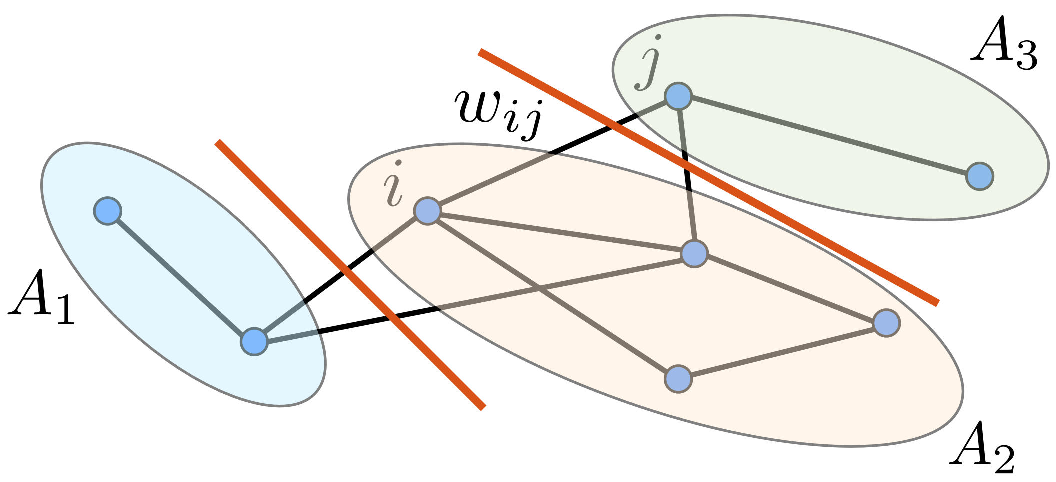


$$NCut(A_1, ..., A_k) = \frac{1}{2} \sum_{i=1}^{k} \frac{W(A_i, \overline{A_i})}{vol(A_i)}$$

Graph clustering (community detection)

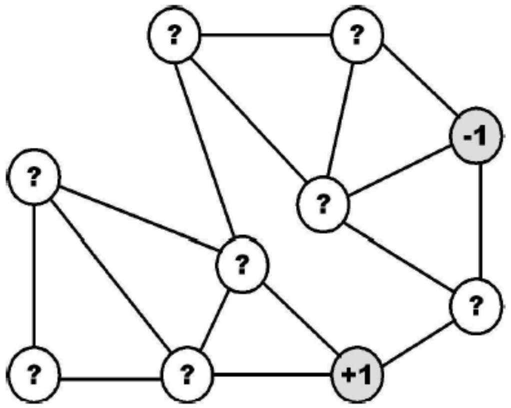


$$NCut(A_1, ..., A_k) = \frac{1}{2} \sum_{i=1}^{k} \frac{W(A_i, \overline{A_i})}{vol(A_i)}$$

- first k eigenvectors of graph Laplacian minimise the graph cut
- eigenvectors of graph Laplacian enable a Fourier-like analysis for graph signals

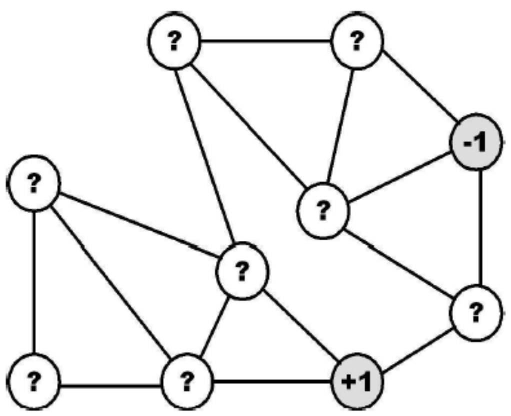
Ulrike von Luxburg, "A tutorial on spectral clustering," Statistics and Computing, 2007.

Semi-supervised learning



Zhu, "Semi-supervised learning with graphs," Ph.D. dissertation, CMU, 2005.

Semi-supervised learning



$$y : \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ +1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\min_{x \in \mathbb{R}^N} ||y - x||_2^2 + \alpha \, x^T L x,$$

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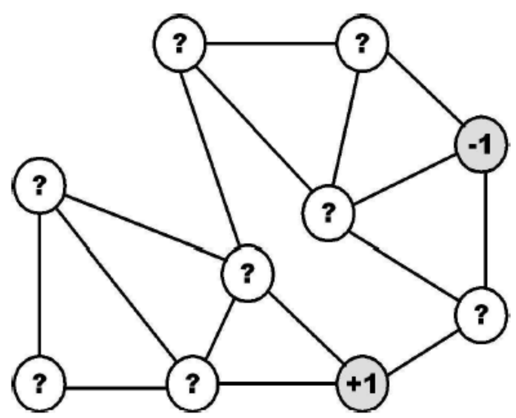
Semi-supervised learning



$$y : \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ +1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

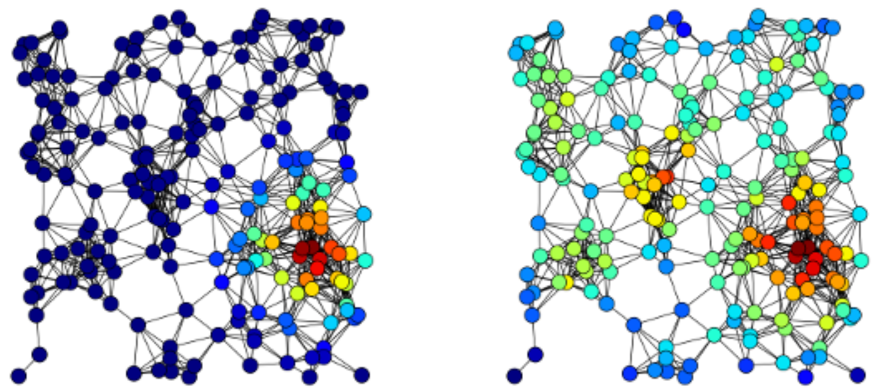
$$\min_{x \in \mathbb{R}^N} ||y - x||_2^2 + \alpha \, x^T L x,$$

- learning by assuming smoothness of predicted labels
- this is equivalent to a denoising problem for graph signal y

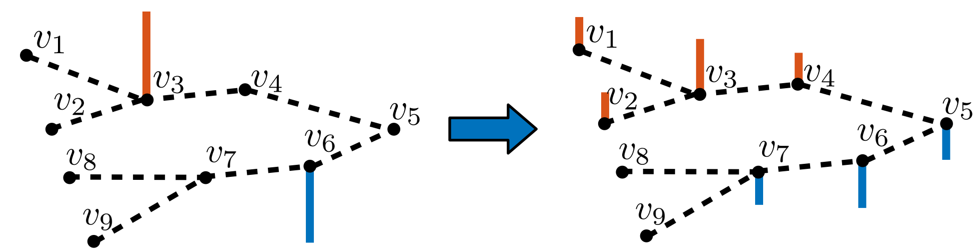
Zhu, "Semi-supervised learning with graphs," Ph.D. dissertation, CMU, 2005.

GSP and the literature

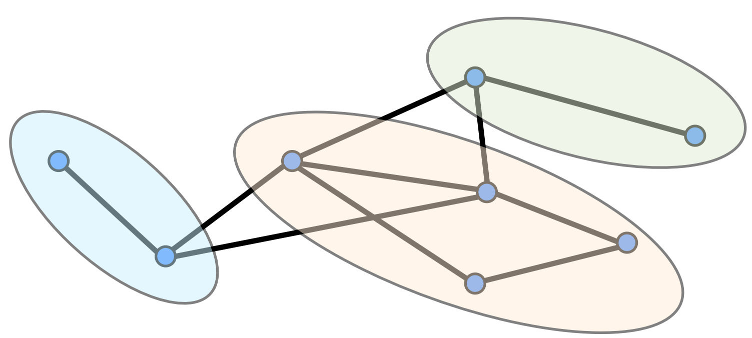
centrality, diffused information, cluster membership, node labels (and node features in general) can ALL be viewed as graph signals



network science



diffusion on graphs



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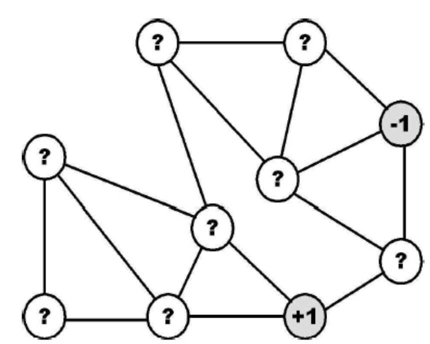
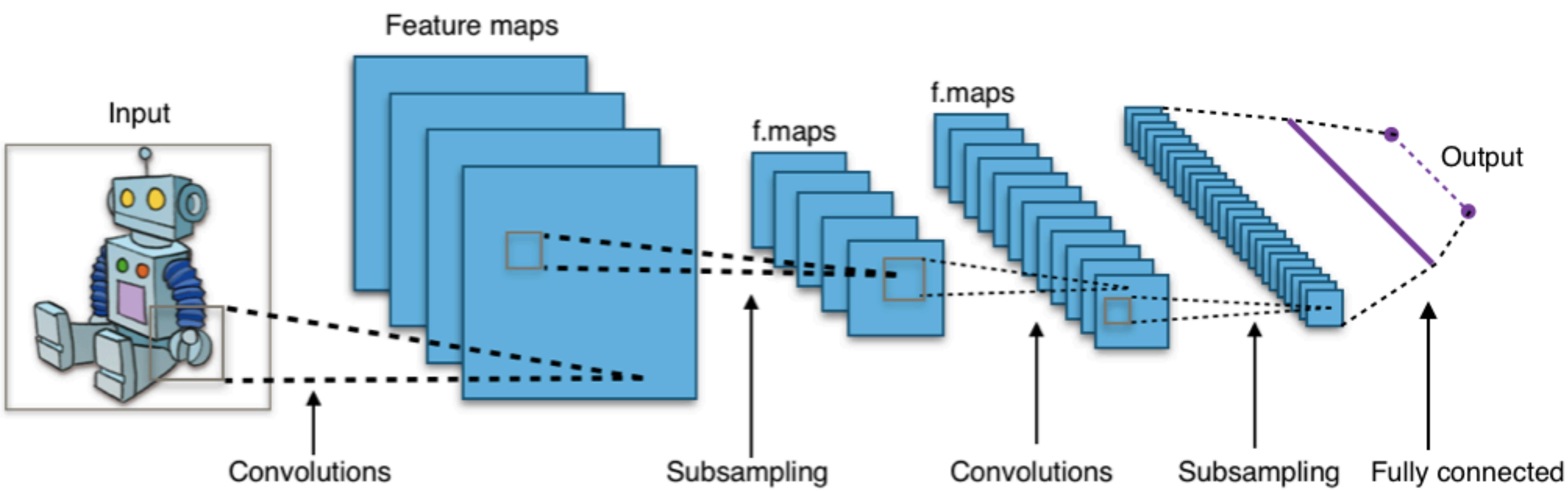


semi-supervised learning

Outline

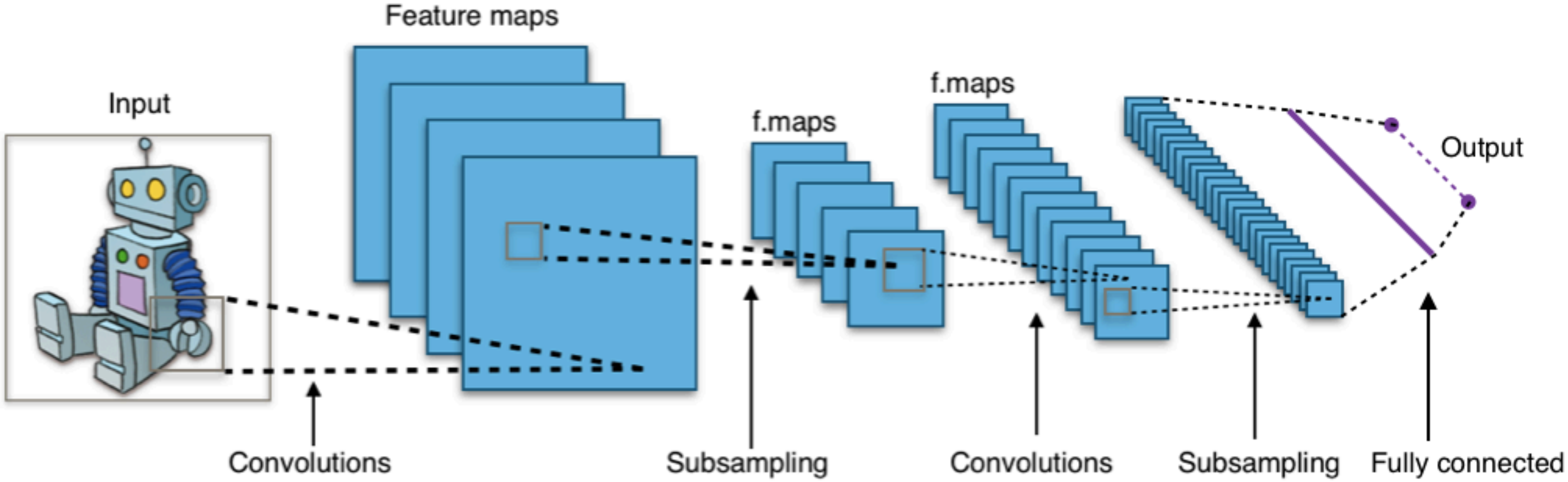
- Graph signal processing (GSP): Basic concepts
- Filtering of graph signals: Basic tool of GSP
- Connection with literature
- Research topics inspired by GSP
- Applications

Deep learning on graphs



https://en.wikipedia.org/wiki/File:Typical_cnn.png

Deep learning on graphs



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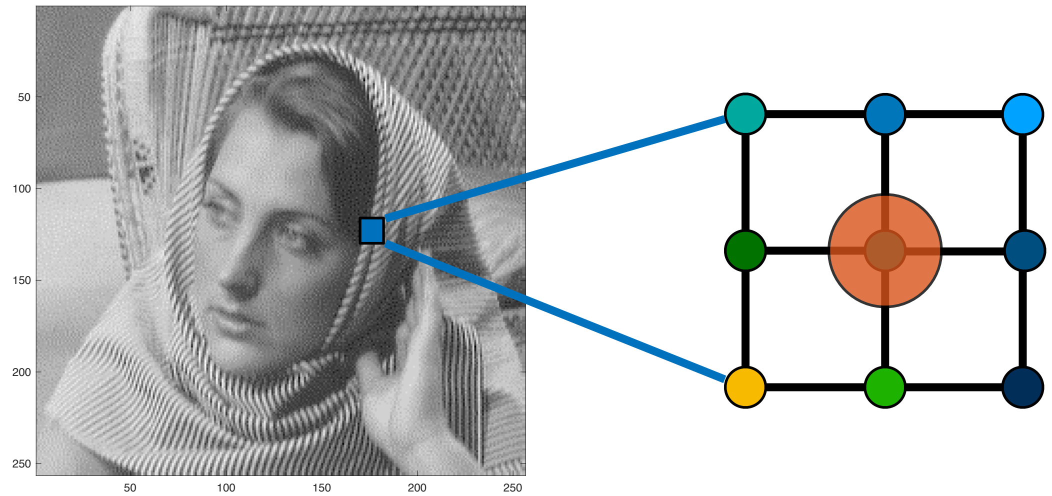
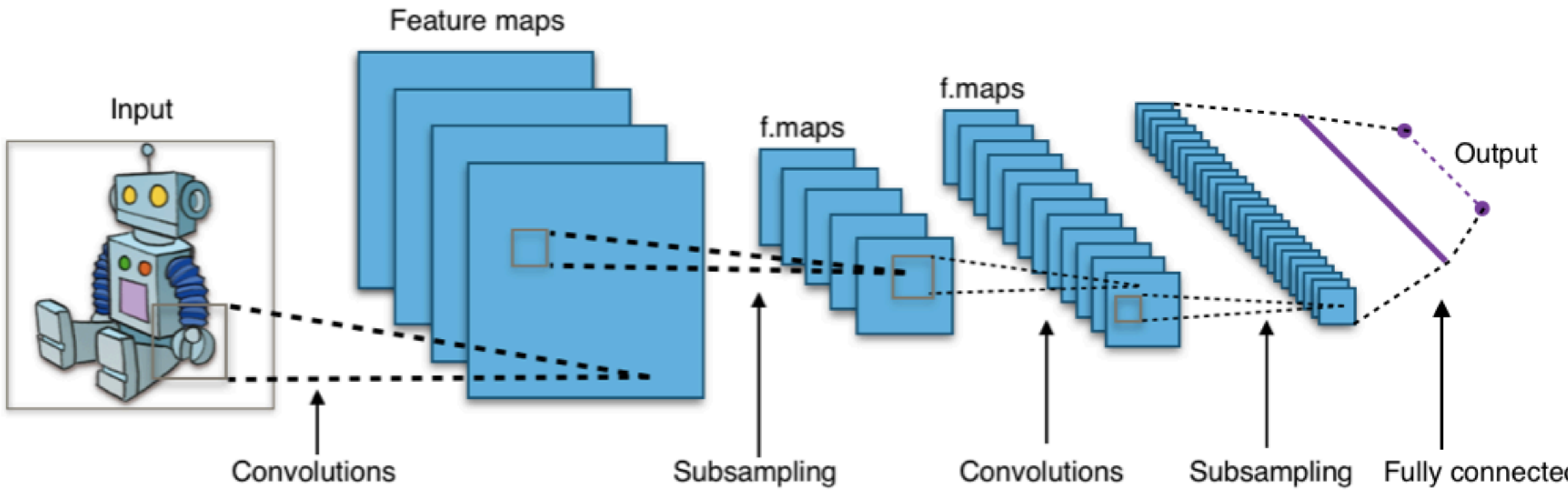


Deep learning on graphs



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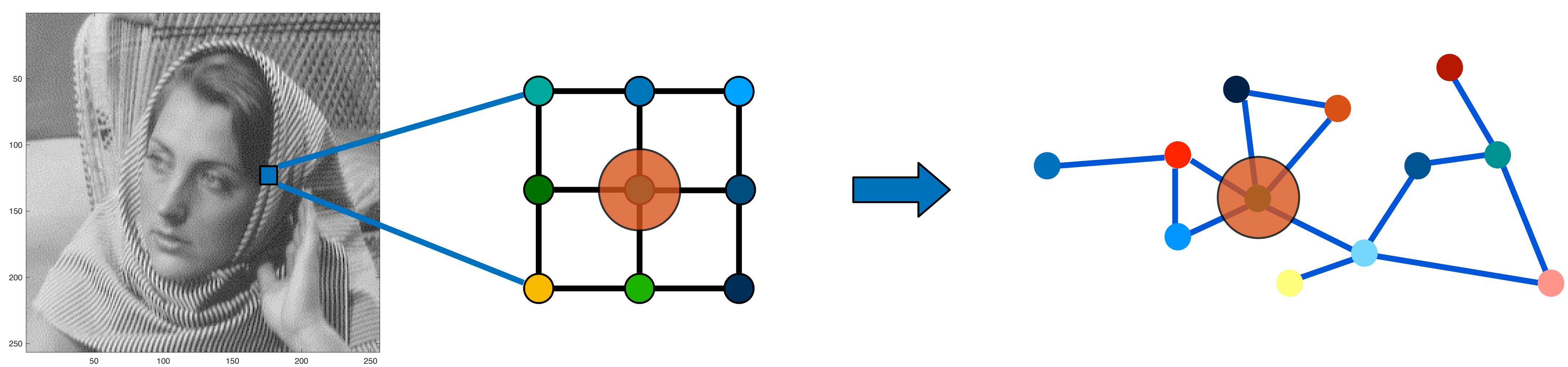
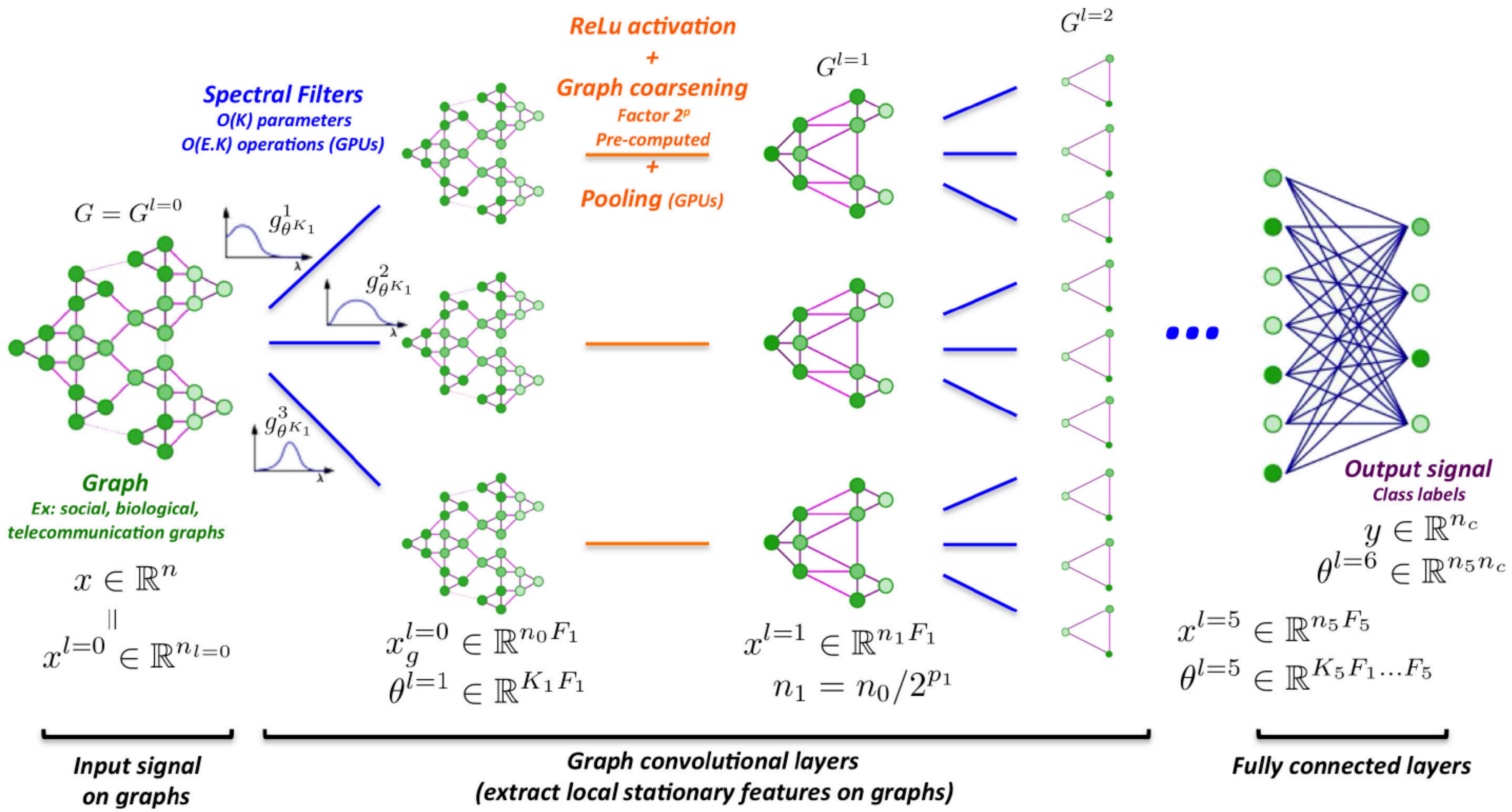
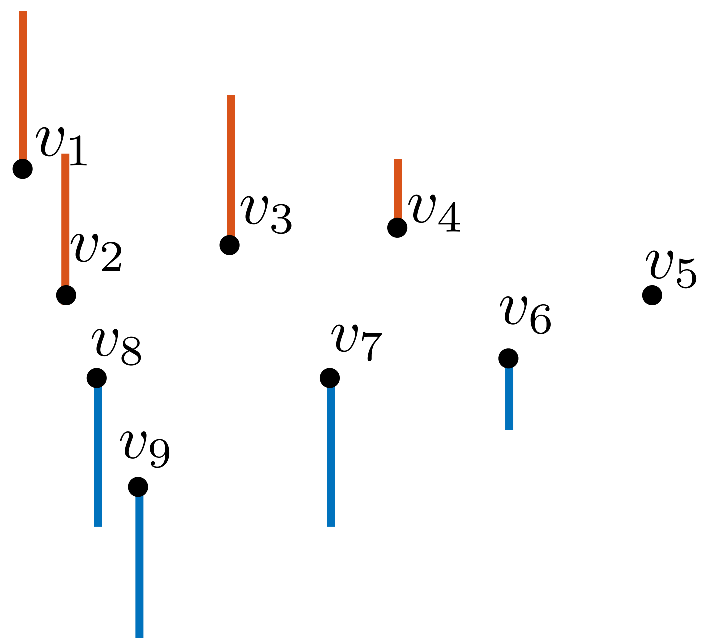


Deep learning on graphs



Defferrard et al., "Convolutional neural networks on graphs with fast localized spectral filtering," NIPS, 2016.

Inferring graph structure from data



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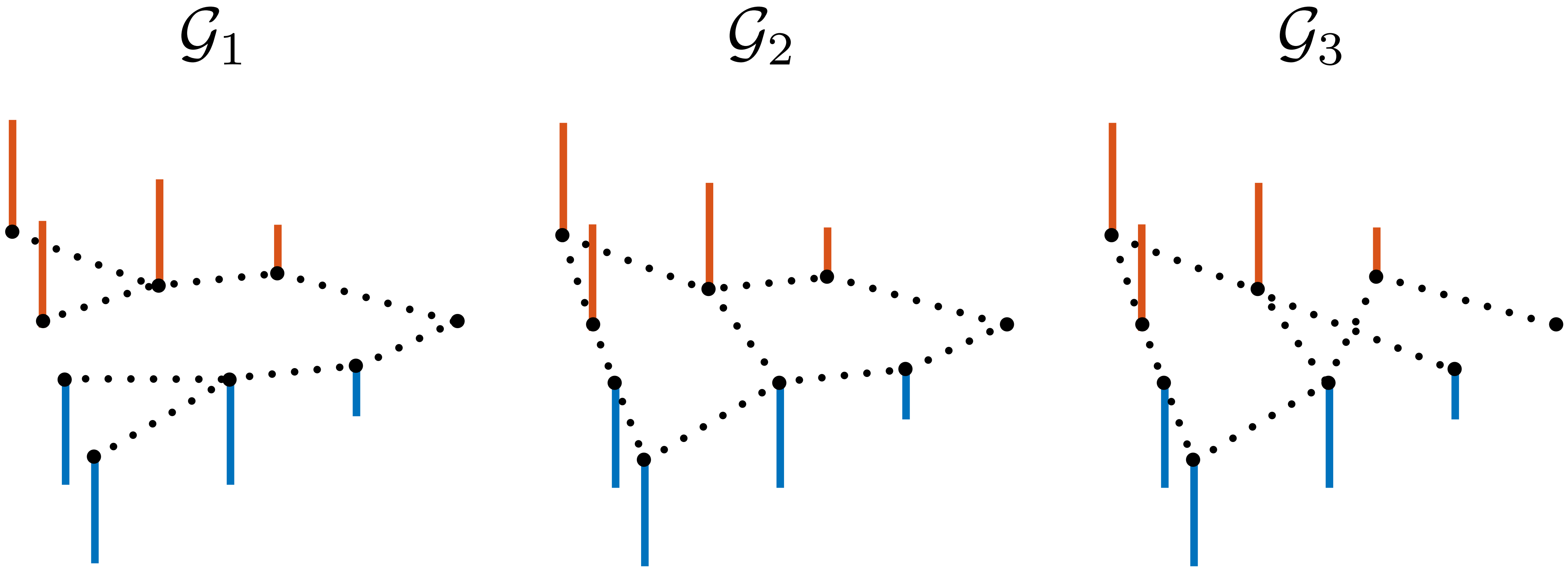
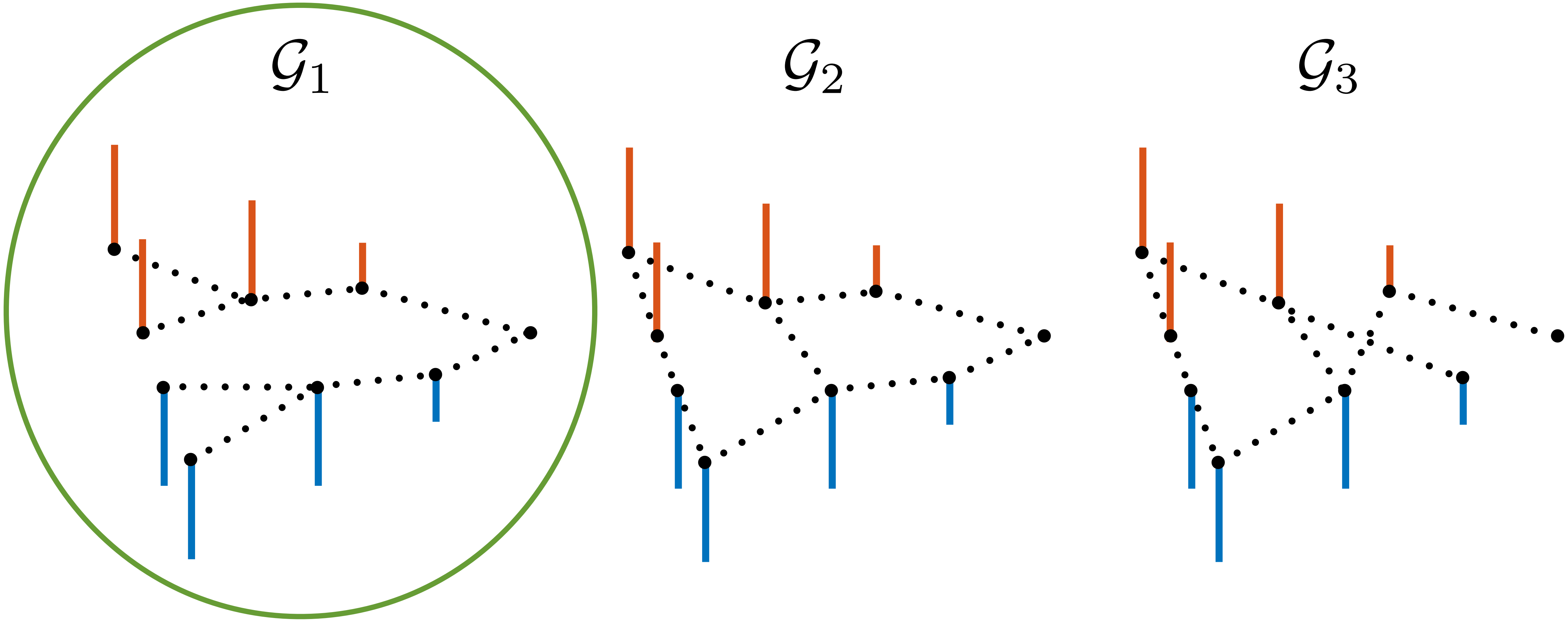


Inferring graph structure from data



idea: choose the structure that enforces certain signal characteristics

Dong et al., "Learning Laplacian matrix in smooth graph signal representations," IEEE TSP, 2016.

Outline

- Graph signal processing (GSP): Basic concepts
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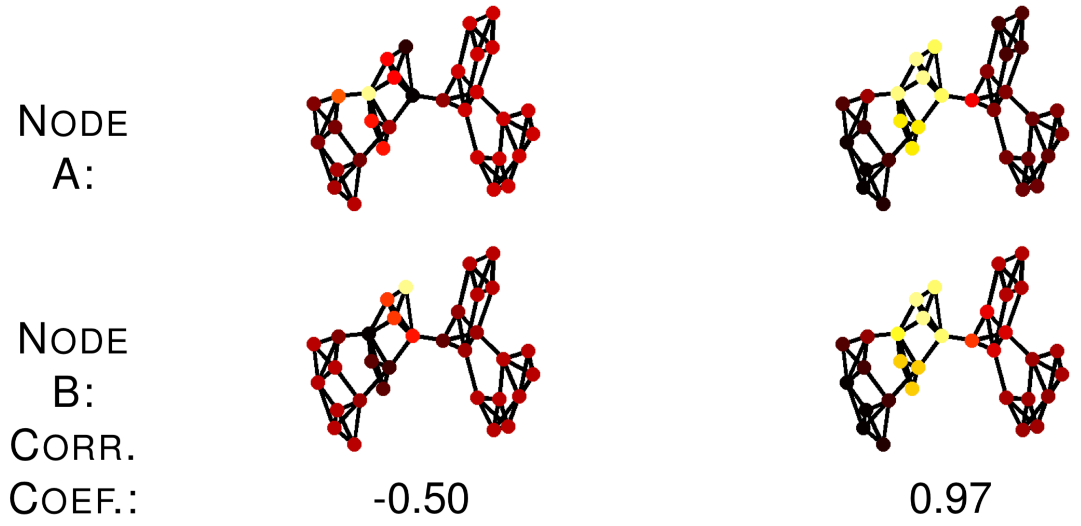
Application I: Community detection

spectral graph wavelets
at different scales:

$$\boldsymbol{D}_s(a,b) = 1 - \frac{\psi_{s,a}^\top \psi_{s,b}}{||\psi_{s,a}||_2 ||\psi_{s,b}||_2}.$$



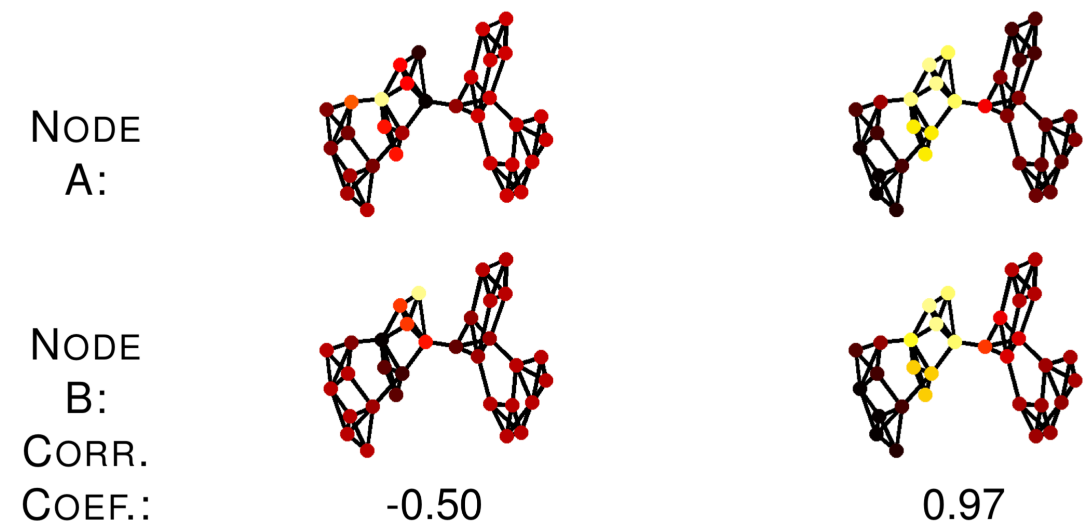
small scale **large scale**

Hammond et al., "Wavelets on graphs via spectral graph theory," Applied and Computational Harmonic Analysis, 2011.
Tremblay and Borgnat, "Graph wavelets for multiscale community mining," IEEE TSP, 2014.

Application I: Community detection

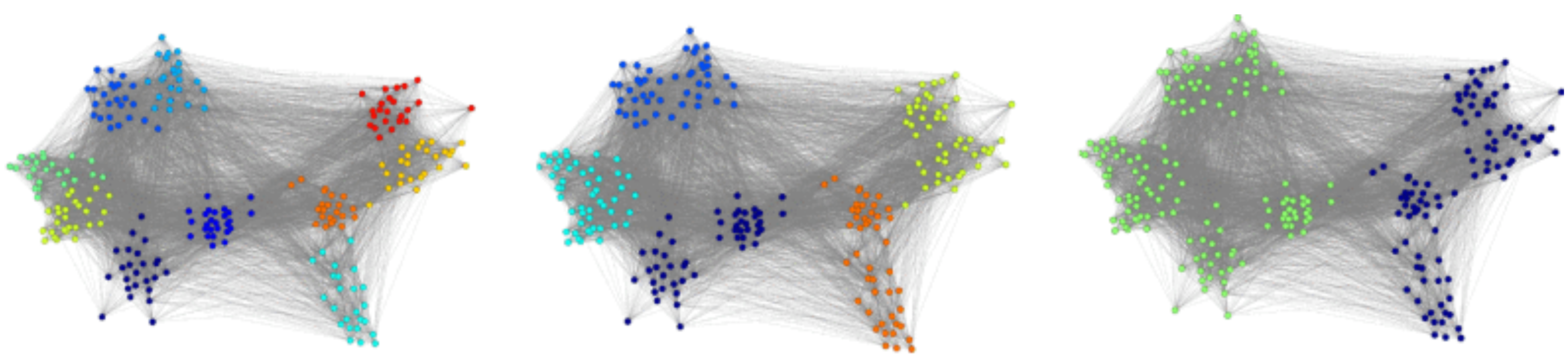
spectral graph wavelets at different scales:

$$\boldsymbol{D}_s(a,b) = 1 - \frac{\psi_{s,a}^{\top}\psi_{s,b}}{||\psi_{s,a}||_2 ||\psi_{s,b}||_2}.$$



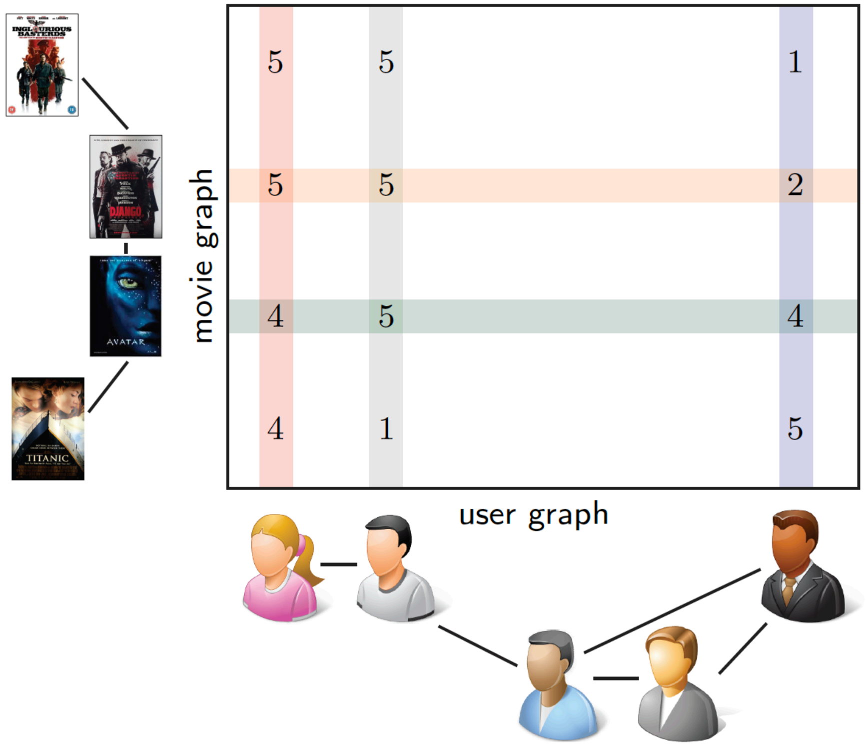
small scale **large scale**

multi-scale community detection:



Hammond et al., "Wavelets on graphs via spectral graph theory," Applied and Computational Harmonic Analysis, 2011.
Tremblay and Borgnat, "Graph wavelets for multiscale community mining," IEEE TSP, 2014.

Application II: Recommender systems



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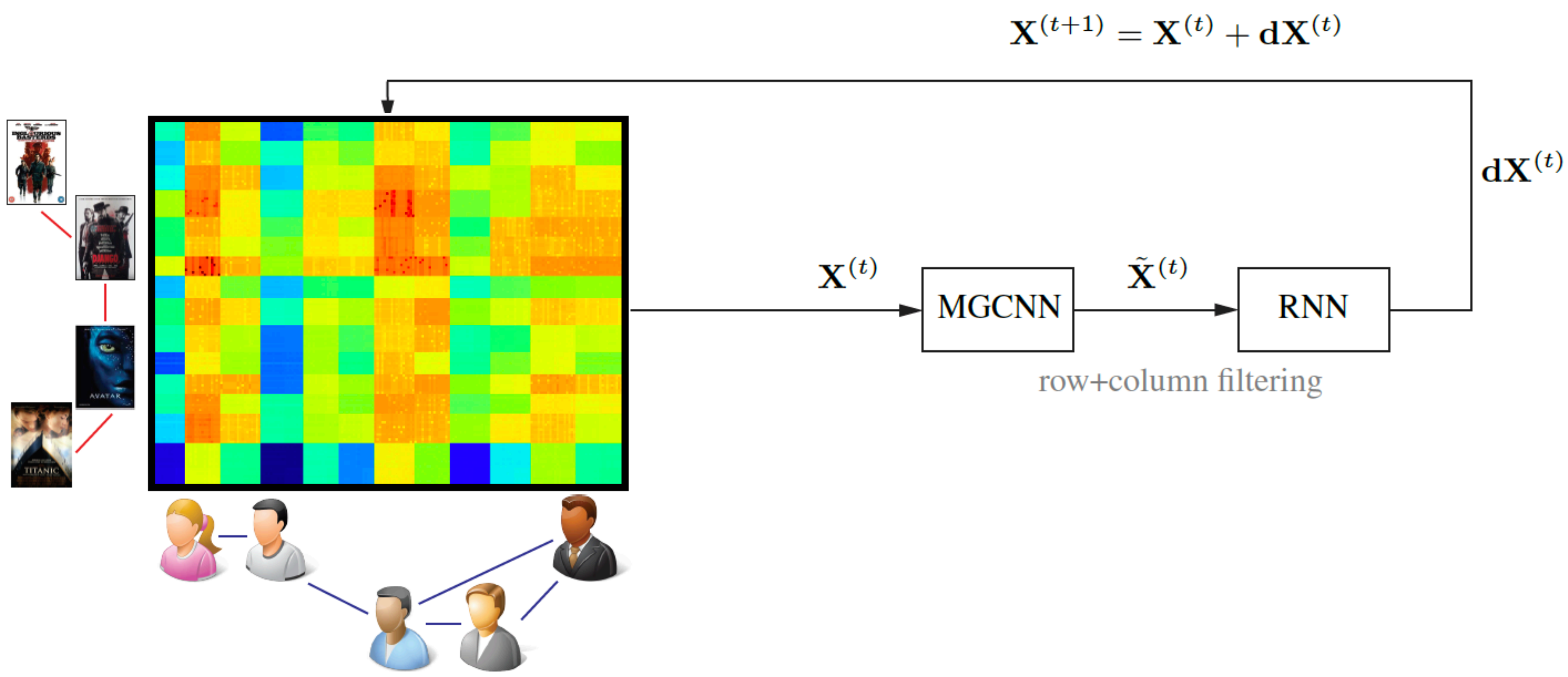
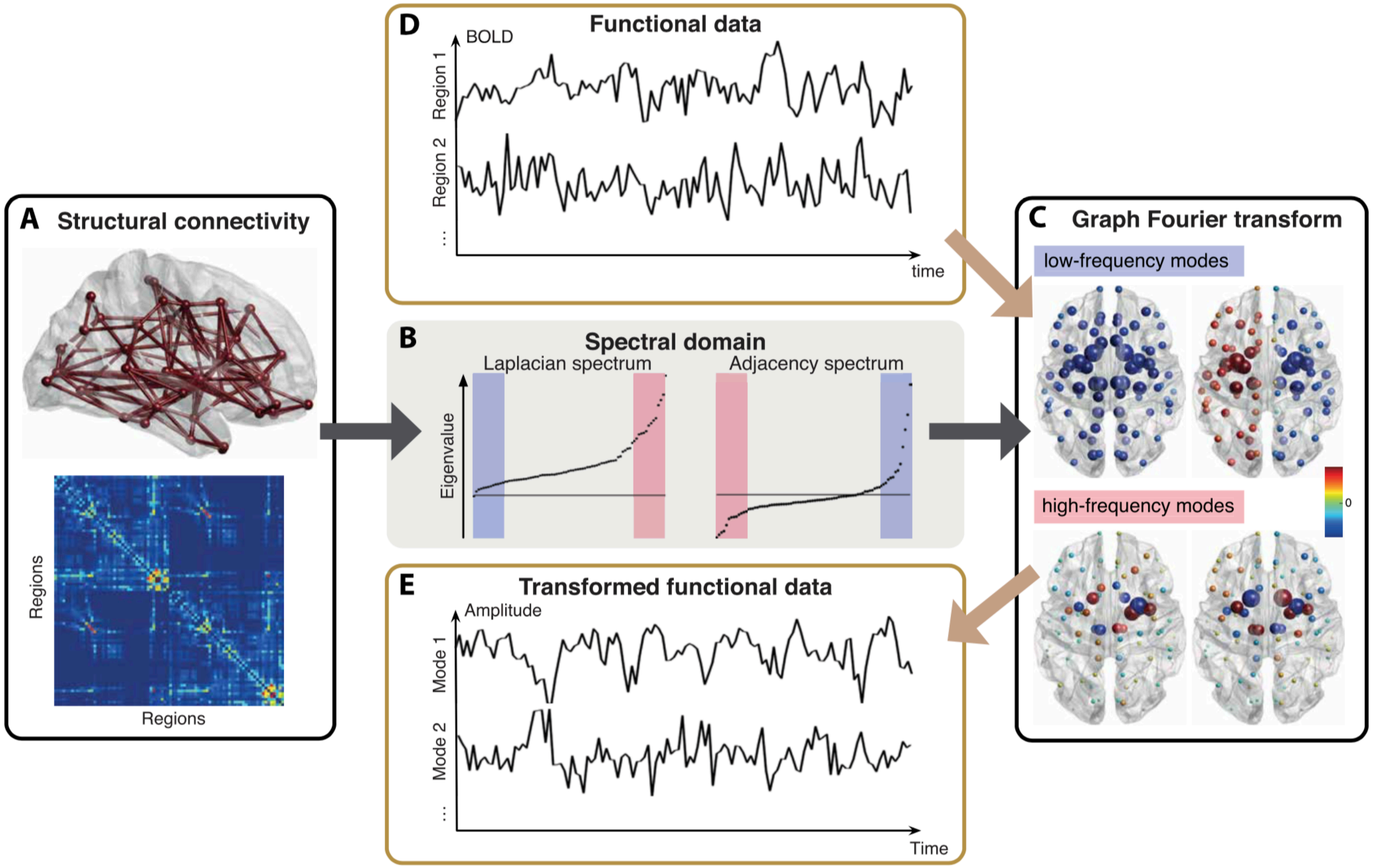
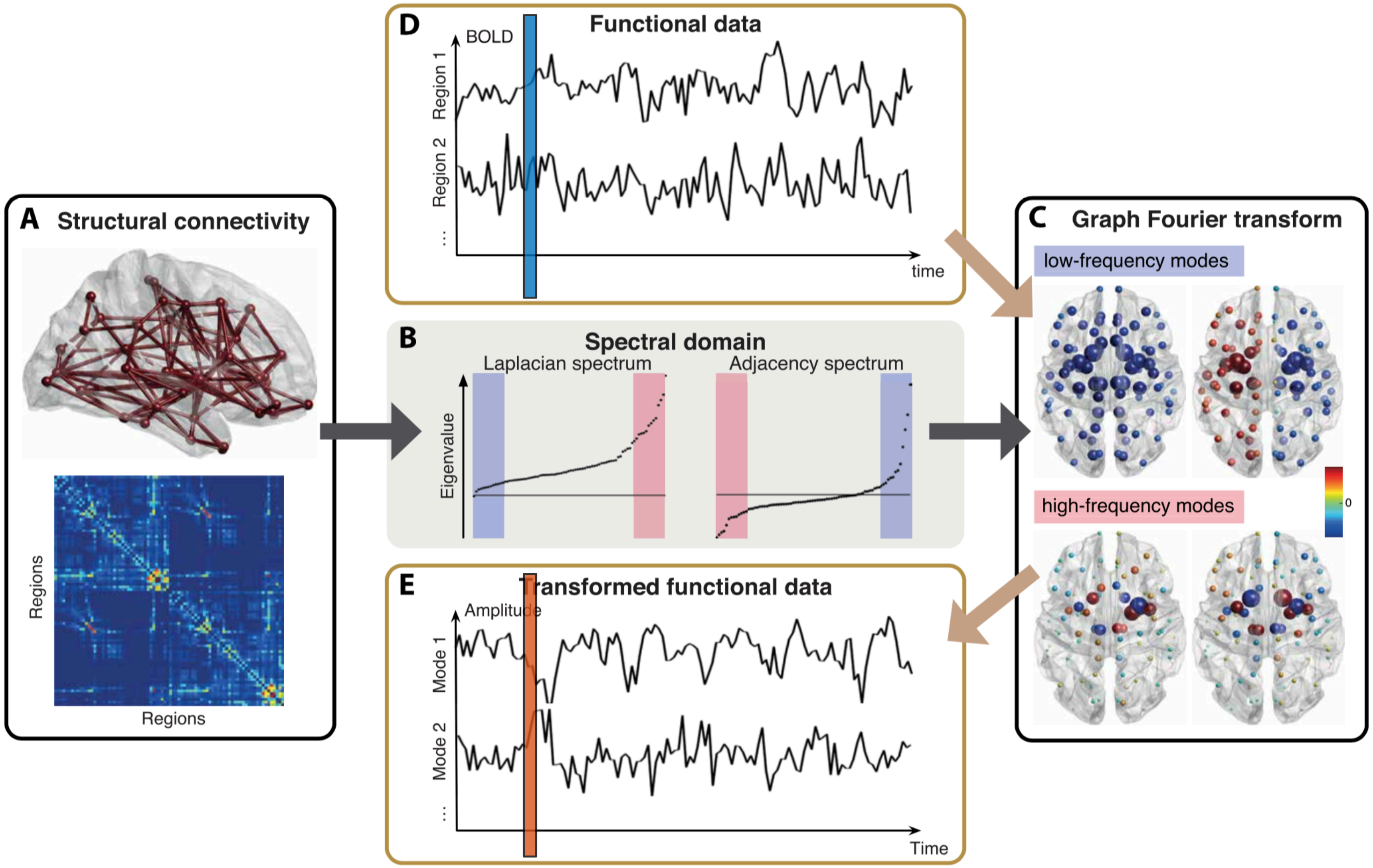


Monti et al., "Geometric matrix completion with recurrent multi-graph neural networks," NIPS, 2017.

Application III: Functional brain imaging

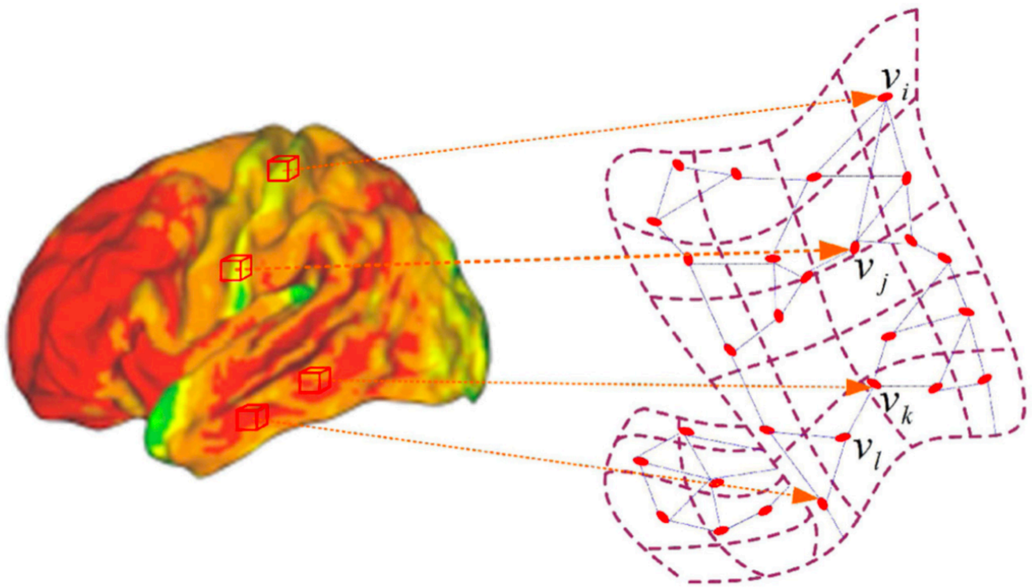


Application III: Functional brain imaging



Huang et al., "A graph signal processing perspective on functional brain imaging," Proceedings of the IEEE, 2018.

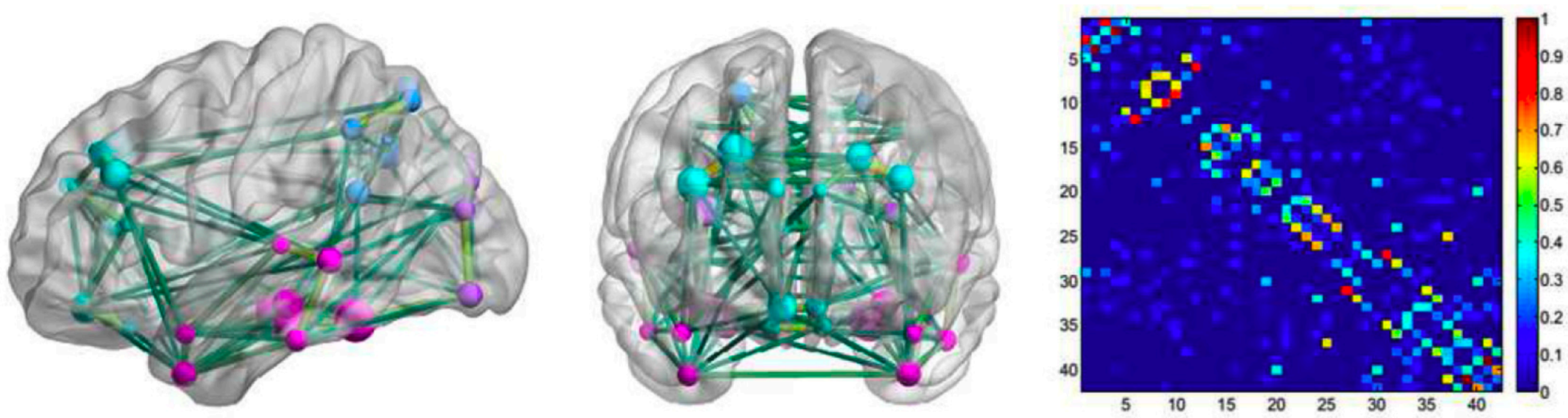
Application IV: Inferring brain connectivity



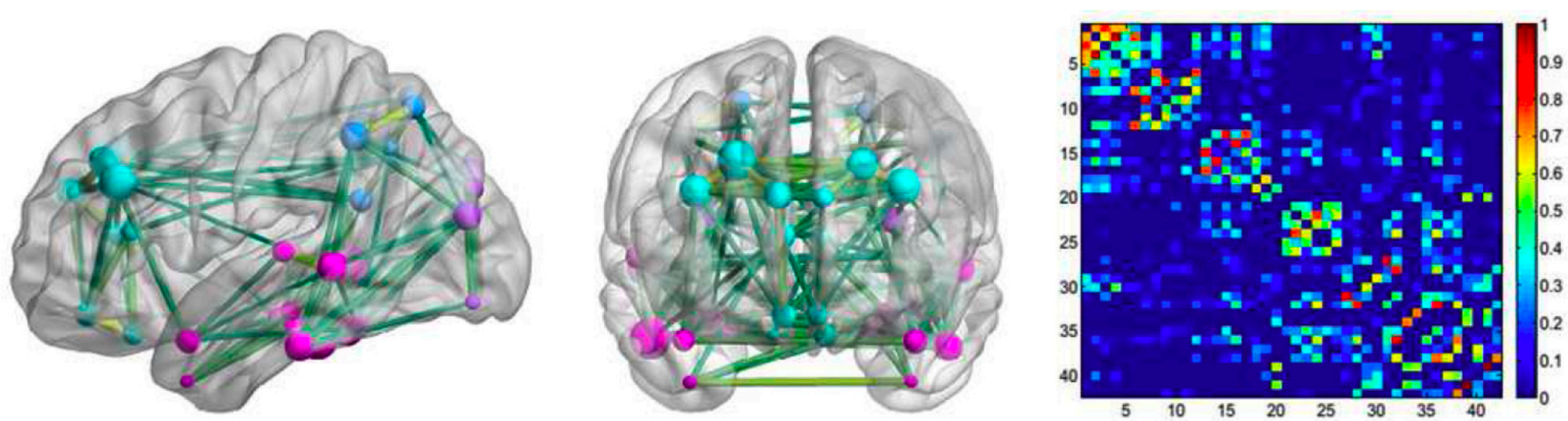
Hu et al., "A spectral graph regression model for learning brain connectivity of Alzheimer's disease," PLOS ONE, 2015.

Application IV: Inferring brain connectivity

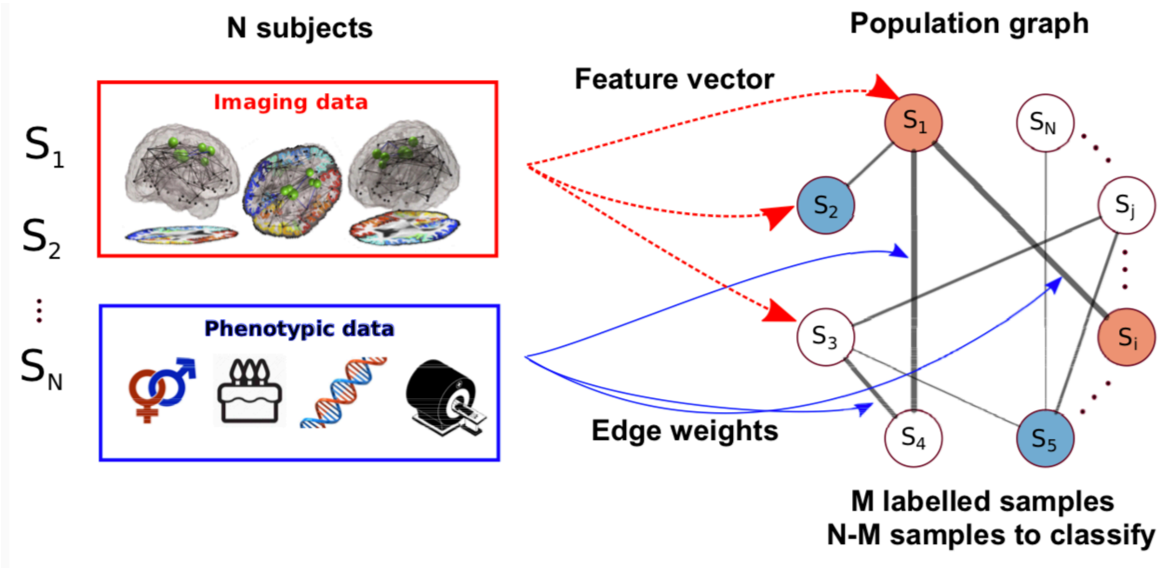
Alzheimer's disease



normal control

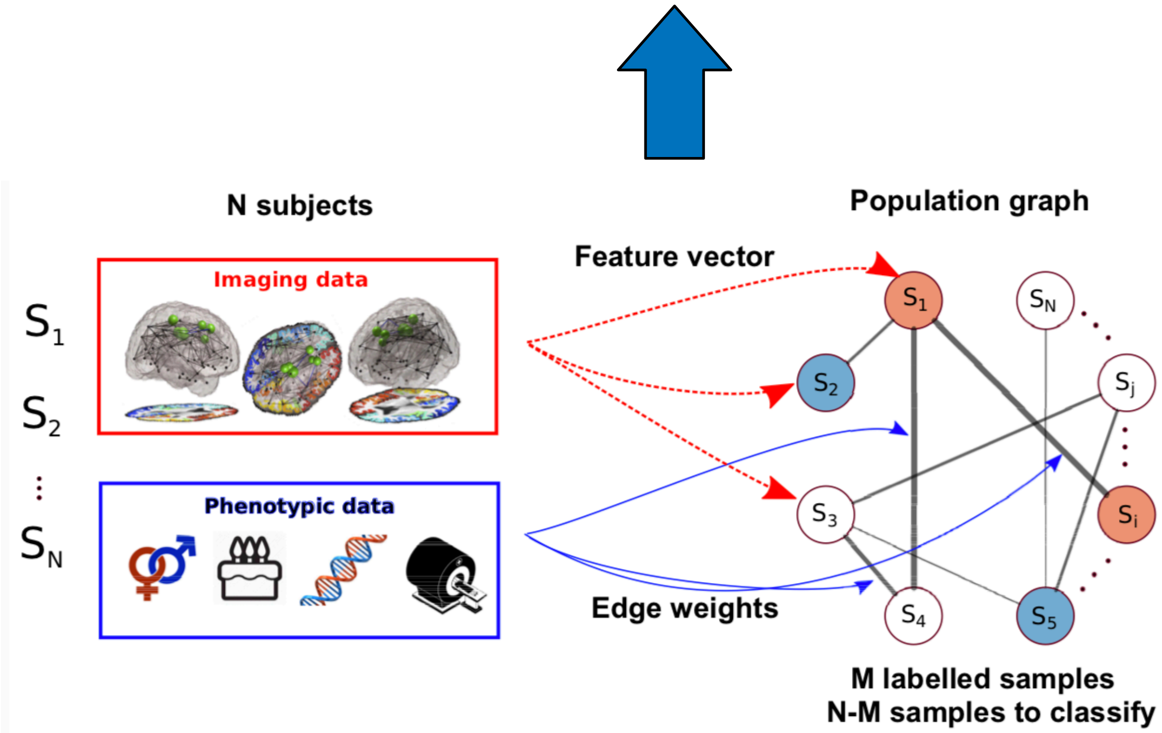


Application V: Disease classification



Application V: Disease classification

features extracted from brain analysis



Application V: Disease classification

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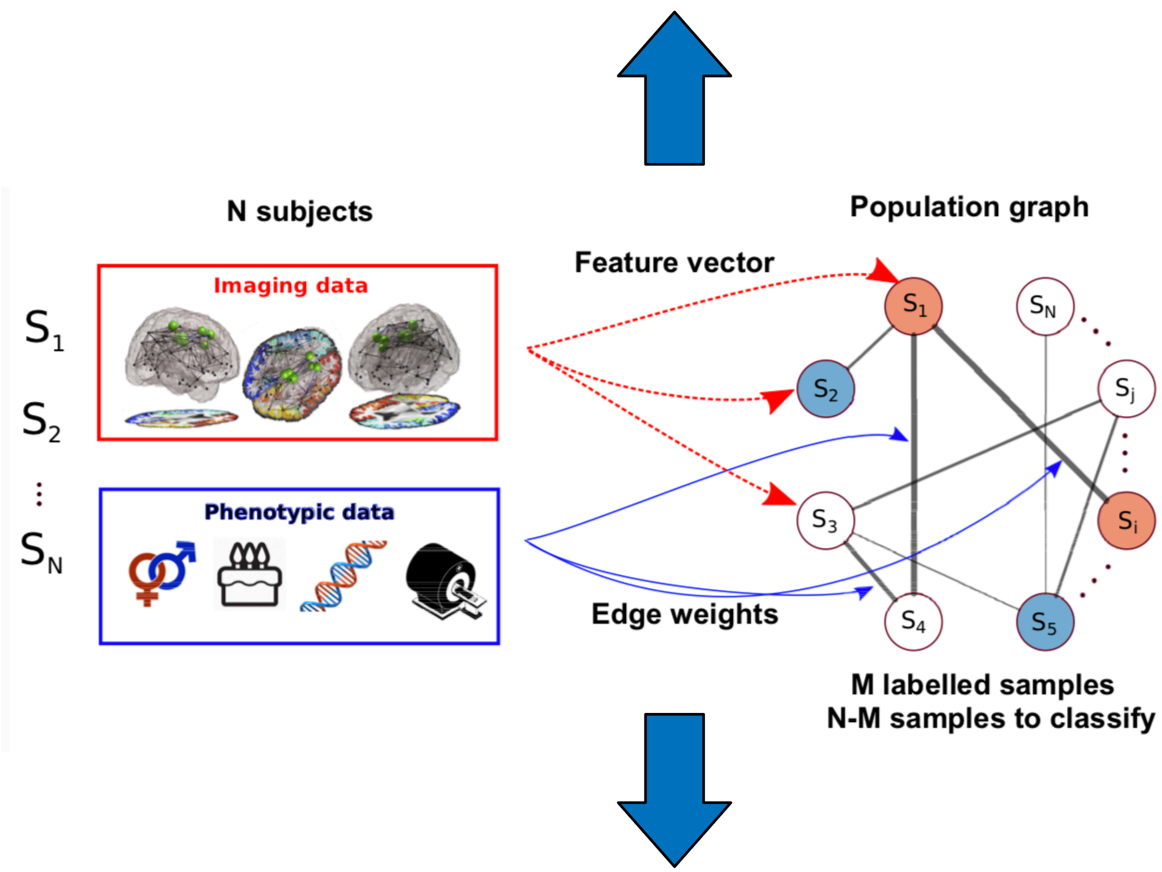
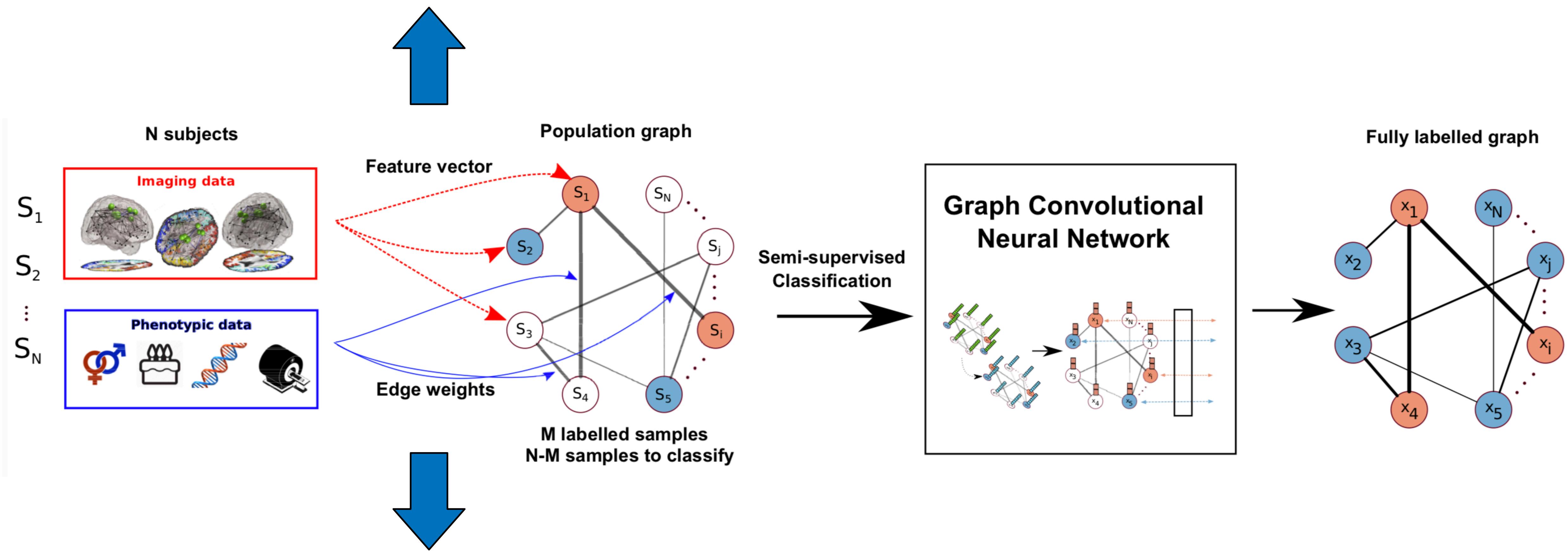


similarity in phenotypic data

Parisot et al., "Disease prediction using graph convolutional networks," Medical Image Analysis, 2018.

Application V: Disease classification

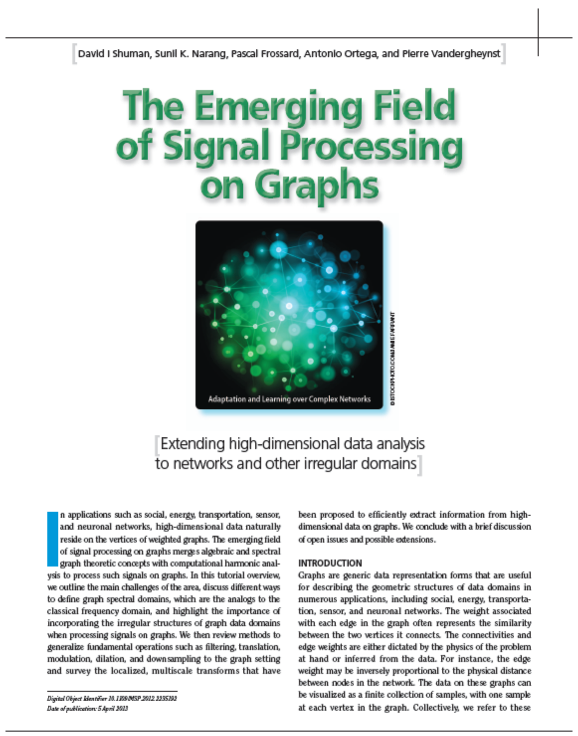
features extracted from brain analysis

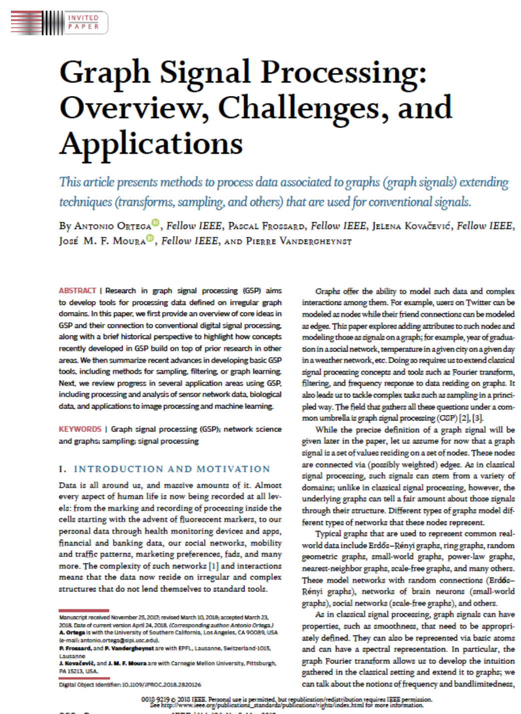


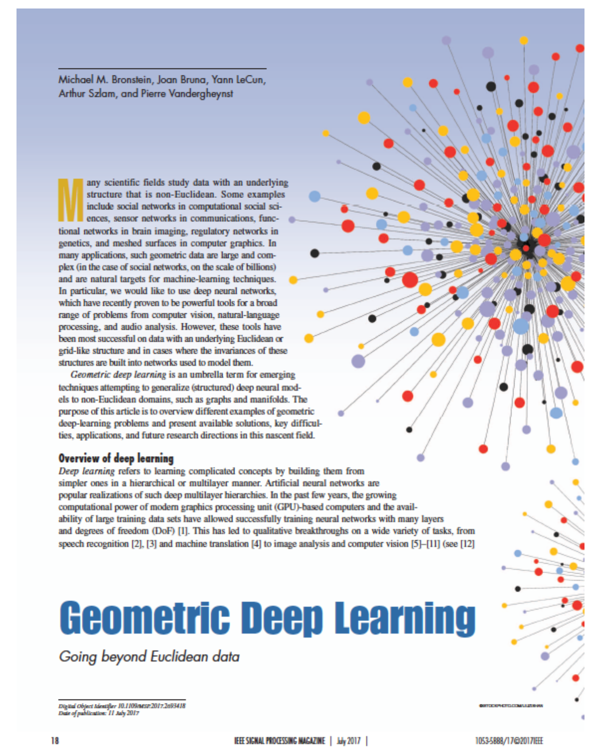
similarity in phenotypic data

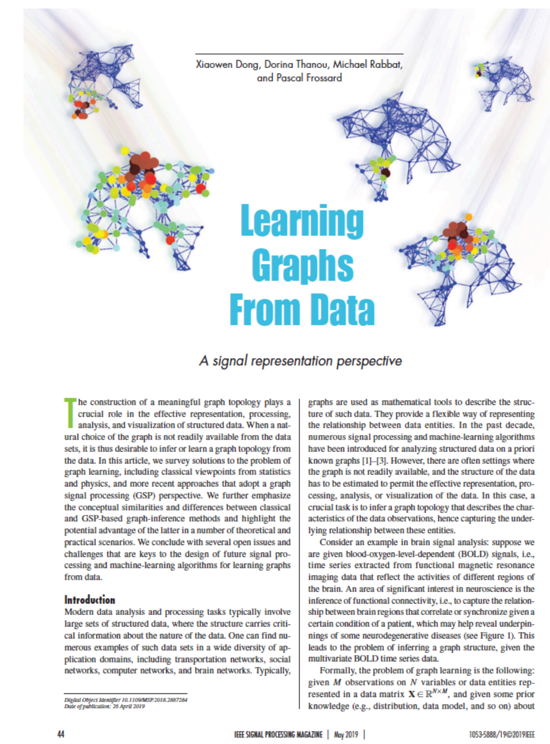
Papers & Resources

- Tutorial/overview papers:









- More available at: http://web.media.mit.edu/~xdong/resource.html