The Complexity of Grid Coverage by Swarm Robotics

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Abstract. In this paper we discuss the task of efficiently using ant-like robotic agents for covering a connected region on the $\mathbf{Z}^2$ grid, whose shape and size are unknown in advance, and which expands at a given rate. This is done using myopic robots, with no ability to directly communicate with each other, where each robot is equipped with only $O(1)$ memory. We show that regardless of the algorithm used, and the robots' hardware and software specifications, the minimal number of robots required in order to enable such coverage is $\Omega(\sqrt{n})$ (where n is the initial size of the connected region). In addition, we show that when the region expands at a sufficiently slow rate, a team of $\Theta(\sqrt{n})$ robots could cover it in at most $O(n^2 \ln n)$ time. Regarding the coverage of non-expanding regions in the grid, we improve the current best known result of $O(n^2)$ by demonstrating an algorithm of worse case completion time of $O(\frac{1}{k}n^{1.5} + n)$, and faster for shapes of perimeter length which is shorter than $O(n)$.

1 Introduction

In this paper we examine a problem in which a group of ant-like robotic agents must cover an unknown region in the grid, that possibly expands over time. This problem is also strongly related to the problem of distributed search after mobile and evading target(s) [3, 4] or the problems discussed under the names of "Lions and Men" pursuits [6]. We analyze such issues using the results presented in [1, 2, 11], concerning the *Cooperative Cleaners* problem, a problem that assumes a regular grid of connected 'tiles', part of which are 'dirty', the 'dirty' tiles forming a connected region of the grid. On this dirty grid region several agents move, each having the ability to 'clean' the place it is located in. In the dynamic variant of this problem a deterministic evolution of the environment in assumed, simulating a spreading *contamination* (or spreading *fire*).

First, we discuss the collaborative coverage of static grids. We demonstrate that the best completion time known to date ($O(n^2)$, achievable for example using the *LRTA** search algorithm) can be improved to guarantee grid coverage in $O(\frac{1}{k}n^{1.5} + n)$ time. Later, we discuss the coverage of expanding domains. We show that using any conceivable algorithm, and using as sophisticated and potent robotic agents as possible, the minimal number of robots below which covering such a region is impossible equals $\Omega(\sqrt{n})$. We then show that when the region expands sufficiently slow (specifically, every $O(\frac{c_0}{\gamma_1})$ time steps, where c_0 is the circumference of the region and where γ_1 is a geometric property of the region, ranging between $O(1)$ and $O(\ln n)$), a group of $\Theta(\sqrt{n})$ robots can successfully cover the region. Furthermore, we demonstrate that in this case a cover time of $O(n^2 \ln n)$ is guaranteed.

2 Related Work

A search for analytic results concerning the completion time of ant-robots covering an area in the grid revealed only a handful of works. The main result in this regard is that of [7], where a swarm of ant-like robots is used to repeatedly cover an unknown area, using a real time search method called *node counting*, using integer markers that are placed on the graph's nodes. The cover time of teams of ant robots that use node counting is shown in [8,9] to be $t_k(n) = O(n^{\sqrt{n}})$, when $t_k(n)$ denotes the cover time of a region of size n using k robots. Another algorithm to be mentioned in this scope is *LRTA**, whose multi-robotics variant is shown in [8] to guarantee cover time of undirected connected graphs in polynomial time. Specifically, on grids it guarantee coverage in $O(n^2)$ time.

Vertex-Ant-Walk, a variant of the node counting algorithm presented in [12], is shown to achieve a coverage time of $O(n\delta_G)$, where δ_G is the graph's diameter, implying a worse-case completion time of $O(n^2)$ in grids. This work is based on a previous work in which a cover time of $O(n^2\delta_G)$ was demonstrated [10].

An algorithm named *Exploration as Graph Construction*, providing a coverage of degree bounded graphs in $O(n^2)$ time, can be found in [5]. Here a group of limited ant robots explore an unknown graph using special "markers".

We next show that the problem of collaborative coverage in static grids can be completed in $O(\frac{1}{k}n^{1.5} + n)$ time and that collaborative coverage of dynamic grids can be achieved in $O(n^2 \ln n)$.

3 The Dynamic Cooperative Cleaners Problem

Following is a short summary of the *Cooperative Cleaners* problem, as appears in [11] (static variant) and [1,2] (dynamic variant). Assuming a discrete time, let the undirected graph $G(V, E)$ denote a two dimensional integer grid $\mathbf{Z}^2$, whose vertices (or "*tiles*") have a binary property called '*contamination*'. Let $cont_t(v)$ state the contamination state of the tile v at time t, taking the values "*on*" or "*off*". Let F_t be the contaminated sub-graph of G at time t, and let F_0 be simply connected. Let a group of k robots that can move on the grid G (moving from a tile to its neighbor in one time step) be placed at time t_0 on F_0, at point $p_0 \in F_t$. Each robot is equipped with a sensor capable of telling the status of all tiles in the digital sphere of diameter 7, which surrounds the robot. Each robot is equipped with a memory of size $O(1)$ bits. When a robot moves to a tile v, it has the possibility of cleaning this tile (i.e. causing $cont(v)$ to become *off*. The robots do not have any prior knowledge of the shape or size of the sub-graph F_0.Every

d time steps, the contamination spreads. That is, if $t = nd$ for some positive integer n, then $\forall v \in F_t \; \forall u \in 4 - Neighbors(v)$, $cont_{t+1}(u) = on$. The robots' goal is to clean G by eliminating the contamination entirely. It is important to note that no central control is allowed, no communication is allowed, and that the system is fully decentralized.

A Survey of Previous Results. The cooperative cleaners problem was previously studied in [1, 2, 11]. A cleaning algorithm called **SWEEP** was proposed (used by a decentralized group of simple mobile robots, for exploring and cleaning an unknown "contaminated" sub-grid F, expanding every d time steps) and its performance analyzed. The **SWEEP** algorithm is based in a constant traversal of the contaminated region, preserving the connectivity of the region while cleaning all *non critical points* — points which when cleaned disconnect the contaminated region. Following are several results that we later use. Note that *cleaning* a region is equivalent to *covering* it, as the number of uncovered tiles is upper bounded by the number of remaining contaminated ones.

Result 1 ***(Cleaning a Non-expanding Contamination)*** *The time it takes for a group of K robots using the **SWEEP** algorithm to clean a region F of the grid is at most:*

$$t_{static} \triangleq \frac{8(|\partial F_0| - 1) \cdot (W(F_0) + k)}{k} + 2k$$

$W(F)$ denotes the depth of the region F (the shortest path from some internal point in F to its boundary, for the internal point whose shortest path is the longest) and ∂F denotes the boundary of F, defined via $\partial F = \{v \mid v \in F \wedge 8-Neighbors(v) \cap (G \setminus F) \neq \emptyset\}$.

Result 2 ***(Universal Lower Bound on Contaminated Area)*** *Using any cleaning algorithm, the area at time t of a contaminated region that expands every d time steps can be recursively lower bounded, as follows :*

$$S_{t+d} \geq S_t - d \cdot k + \left\lfloor 2\sqrt{2 \cdot (S_t - d \cdot k) - 1} \right\rfloor$$

Here S_t denotes the area of the contaminated region at time t (such that $S_0 = n$).

Result 3 ***(Upper Bound on Cleaning Time for SWEEP on Expanding Domains)*** *For a group of k robot using the **SWEEP** algorithm to clean a region F on the grid, that expands every d time steps, the time it takes the robots to clean F is at most d multiplied by the minimal positive value of the following two numbers :*

$$\frac{(A_4 - A_1 A_3) \pm \sqrt{(A_1 A_3 - A_4)^2 - 4 A_3 (A_2 - A_1 - A_1 A_4)}}{2 A_3}$$

where :

$$A_1 = \frac{c_0 + 2 - \gamma_2}{4} \quad , \quad A_2 = \frac{c_0 + 2 + \gamma_2}{4} \quad , \quad A_3 = \frac{8 \cdot \gamma_2}{d \cdot k} \quad , \quad A_4 = \gamma_1 - \frac{\gamma_2 \cdot \gamma}{d} \quad ,$$

$$\gamma_1 = \psi\left(1 + A_2\right) - \psi\left(1 + A_1\right) \quad , \quad \gamma_2 = \sqrt{(c_0+2)^2 - 8S_0 + 8} \quad ,$$
$$\gamma = \frac{8(k + W(F_0))}{k} - \frac{d - 2k}{|\partial F_0| - 1}$$

Here c_0 is the circumference of the initial region F_0, and where $\psi(x)$ is the *Digamma* function — the logarithmic derivative of the *Gamma function*.

4 Grid Coverage — Analysis

We first present the cover time of a group of robots operating in non-expanding domains, using the **SWEEP** algorithm.

Theorem 1. *Given a connected region of $S_0 = n$ tiles and perimeter c_0, then k ant-like robots can cover it using $O\left(\frac{1}{k}S_0^{1.5} + S_0\right)$ time.*

Proof. *Since $|\partial F_0| = \Theta(c_0)$, and $W(F_0) = O(\sqrt{S_0})$, recalling Result 1 we see that :*

$$t_k(n) = t_{static}(k) = O\left(\frac{1}{k}\sqrt{S_0}\cdot c_0 + c_0 + k\right)$$

As $c_0 = O(S_0)$ and as for practical reasons we assume that $k < n$ this equals :

$$t_k(n) = t_{static}(k) = O\left(\frac{1}{k}S_0^{1.5} + S_0\right)$$

We now examine the problem of covering expanding domains. The lower bound for the number of robots required for completing is as follows.

Theorem 2. *Given a region of size $S_0 \geq 3$ tiles, expanding every d time steps, then a team of less than $\frac{\sqrt{S_0}}{d}$ robots cannot clean the region, regardless of the algorithm used.*

Proof. *Recalling Result 2, and by assigning $k = \frac{\sqrt{S_0}}{d}$ we can see that :*

$$\Delta S_t = S_{t+d} - S_t \geq \left\lfloor 2\sqrt{2\cdot(S_t - \sqrt{S_0}) - 1}\right\rfloor - \sqrt{S_0}$$

For any $S_0 \geq 3$, we see that $\Delta S_0 > 0$. In addition, for every $S_0 \geq 3$ we can see that $\frac{dS_t}{dt} > 0$ for every $t \geq 0$. Therefore, for every $S_0 \geq 3$ the size of the region will be forever growing.

Corollary 1. *Given a region of size S_0 tiles, expanding every d time steps, where $d = O(1)$ w.r.t S_0, then a team of less than $\Omega(\sqrt{S_0})$ robots cannot clean the region, regardless of the algorithm used.*

Theorem 3. *Let F be a region of size S_0 tiles, expanding every d time steps. A team of k robots located at $t = 0$ on the same tile cannot clean F, regardless of the algorithm used, if $d^2k < \Omega(R(F))$, where $R(F)$ is the perimeter of the bounding rectangle of F.*

Proof. *For every $v \in F$ let $l(v)$ denote the maximal distance between v and any of the tiles of F, namely :*

$$l(v) = \max\{d(v,u)|u \in F\}$$

Let $C(F) = l(v_c)$ such that $v_c \in F$ is the tile with minimal value of $l(v)$.

Let v_s denote the tile the agents are located in at $t = 0$. Let $v_d \in F$ denote some contaminated tile such that $d(v_s, v_d) = l(v_s)$. Regardless of the algorithm used by the agents, until some agent reaches v_d there will pass at least $l(v_s)$ time steps. Let us assume w.l.o.g that v_d is located to the right (or of the same horizontal coordinate) and to the top (or of the same vertical coordinate) of v_s. Then by the time some agent is able to reach v_d there exists an upper-right quarter of a digital sphere of radius $\left\lfloor \frac{l(v_s)}{d} \right\rfloor + 1$, whose center is v_d. The number of tiles in such a quarter of digital sphere equals :

$$\frac{1}{2}\left\lfloor \frac{l(v_s)}{d} \right\rfloor^2 + \frac{3}{2}\left\lfloor \frac{l(v_s)}{d} \right\rfloor + 1 = \Theta\left(\frac{l(v_s)^2}{d^2}\right)$$

It is obvious that the region cannot be cleaned until v_d is cleaned. Let t_d denote the time at which the first agent reaches v_d. It is easy to see that $t_d \geq l(v_s)$. Therefore, at time t_d there are k agents that has to clean a region of at least $\Theta(\frac{l(v_s)^2}{d^2})$ tiles, spreading every d time steps. Using Theorem 2 we know that k agents cannot clean an expanding region of $k = \frac{\sqrt{S_0}}{d}$ tiles. Namely, at time t_d the k agents could not clean the contaminated tiles if :

$$d^2 k < \Omega\left(l(v_s)\right)$$

As $l(v_s) \geq C(F)$ we know that k agents could not clean an expanding contaminated region where : $d^2 k < \Omega\left(C(F)\right)$. It is easy to see that for every region F, if $R(F)$ is the length of the perimeter of the bounding rectangle of F then $C(F) = \Theta(R(F))$.

Lemma 1. *For every connected region of size $S_0 \geq 3$ and perimeter of length c_0 :*

$$\frac{1}{2}c_0 < \gamma_2 < c_0$$

Proof. *let us assume by contradiction that $(c_0 + 2)^2 \leq (8S_0 + 8)$, implying $c_0 \leq \sqrt{8S_0 + 8} - 2$. However, the minimal circumference of a region of size S_0 is achieved when the region is arranged as an 8-connected digital sphere, in which $c_0 \geq 4\sqrt{S_0} - 4$, contradicting the assumption that $c_0 \leq \sqrt{8S_0 + 8} - 2$ for $S_0 > 5$ and hence, $\gamma_2 \in \mathbb{R}$.*

Let us assume by contradiction that $\gamma_2 < \frac{1}{2}c_0$. This in turn implies :

$$c_0 < -\frac{16}{6} + \sqrt{10\frac{2}{3}S_0 - 8\frac{8}{9}} < 3.266\sqrt{S_0} - 2$$

We know that $c_0 \geq 4\sqrt{S_0} - 4$*, which contradicts the assumption that* $\gamma_2 < \frac{1}{2}c_0$ *for every* $S_0 \geq 3$*. Let us assume by contradiction that* $\gamma_2 > c_0$*, implying that* $c_0 > 4S_0 - 6$*.*

We know that $c_0 \leq 2S_0 - 2$ *(as* c_0 *is maximized when the tiles are arranged in the form of a straight line), contradicting the assumption that* $\gamma_2 > c_0$ *for every* $S_0 \geq 3$*.*

Lemma 2. *For every connected region of size* $S_0 \geq 3$ *and perimeter of length* c_0 *:*

$$\Omega(1) < \gamma_1 < O(\ln n)$$

Proof. *Let us observe* $\gamma_1 : \gamma_1 \triangleq \psi\left(1 + \frac{c_0+2+\gamma_2}{4}\right) - \psi\left(1 + \frac{c_0+2-\gamma_2}{4}\right)$*. From Lemma 1 we can see that* $1 < \left(1 + \frac{c_0+2-\gamma_2}{4}\right) < \frac{1}{4}c_0$*. Note that* $\psi(1) = -\hat{\gamma}$ *where* $\hat{\gamma}$ *is the* Euler-Mascheroni *constant which equals approximately* 0.57721*. In addition,* $\psi(x)$ *is monotonically increasing for every* $x > 0$*. As* $\psi(x)$ *is upper bounded by* $O(\ln x)$ *for large values of* x*, we see that :*

$$-0.58 < \psi(1 + \frac{c_0 + 2 - \gamma_2}{4}) < O(\ln n) \qquad (1)$$

From Lemma 1 we also see that $1 < \left(1 + \frac{c_0+2+\gamma_2}{4}\right) < \frac{1.5}{4}c_0$ *meaning that :*

$$\psi(1 + \frac{c_0 + 2 + \gamma_2}{4}) = \Theta(\ln n) \qquad (2)$$

Combining equations 1 and 2 the rest is implied.

Theorem 4. *Result 3 returns a positive real number for the covering time of a region of* S_0 *tiles that expands every* d *time steps, when the number of robots equals* $\Theta(\sqrt{S_0})$ *and* $d = \Omega(\frac{c_0}{\gamma_1})$*, where* γ_1 *defined in Result 3 shifts from* $O(1)$ *to* $O(\ln S_0)$ *as* c_0 *grows from* $O(\sqrt{S_0})$ *to* $O(S_0)$*.*

Proof. *In order for Result 3 to yield a real number all the following must hold :*

$$d \cdot k \neq 0 \quad , \quad |\partial F| > 1 \quad , \quad A_3 \neq 0 \quad , \quad (c_0 + 2)^2 > 8S_0 - 8$$

$$(A_1A_3 - A_4)^2 \geq 4A_3(A_2 - A_1 - A_1A_4)$$

The first and second requirements hold for every non trivial scenario. The third requirement is implied by the fourth. The fourth assumption is a direct result of Lemma 1.

As for the last requirement, we ask that $A_1^2A_3^2 + A_4^2 \geq 4A_2A_3 - 4A_1A_3 - 2A_1A_3A_4$*, which subsequently means that we must have :*

$$\frac{\gamma_2^2}{d^2k^2}\left(c_0^2 + \gamma_2^2 - c_0\gamma_2\right) + \gamma_1^2 + \frac{\gamma_2^2\gamma^2}{d} - \frac{\gamma\gamma_1\gamma_2}{d} \geq \frac{\gamma_2}{dk} \cdot O\begin{pmatrix} \gamma_2 - c_0\gamma_1 + c_0\frac{\gamma_2\gamma}{d} \\ -\gamma_1 + \gamma_1\gamma_2 - \frac{\gamma_2^2\gamma}{d} \end{pmatrix}$$

Using Lemma 1 and Lemma 2 we should make sure that :

$$\frac{\gamma_2^2}{dk^2}c_0^2 + \gamma_1^2 d + \gamma_2^2\gamma^2 - \gamma\gamma_1\gamma_2 \geq O\left(\frac{c_0\gamma_2\gamma_1}{k} + \frac{c_0\gamma_2^2\gamma}{dk}\right)$$

Using $W(F) = O(\sqrt{S_0})$ and $\Omega(\sqrt{S_0}) = |\partial F| = O(S_0)$ and dividing by γ_2^2 (which we know to be larger than zero), we can write the above as follows :

$$\frac{c_0^2}{dk^2} + \frac{k^2 + d\ln^2 S_0}{c_0^2} + 1 \geq O\left(\frac{\ln S_0}{c_0} + \frac{k\ln S_0}{c_0^2} + \frac{\ln S_0}{k} + \frac{c_0\sqrt{S_0}}{dk^2} + \frac{c_0}{dk} + \frac{1}{d}\right)$$

As $c_0 \geq \sqrt{S_0}$ then $\frac{c_0^2}{dk^2} \geq \frac{c_0\sqrt{S_0}}{dk^2}$. In addition, $1 \geq \frac{1}{d}$ and also $1 \geq \frac{\ln S_0}{c_0}$ and $1 \geq \frac{\ln S_0}{k}$. In order to have also $1 \geq \frac{c_0}{dk}$ we must have : $d \cdot k = \Omega(c_0)$

In addition, we also require that the cleaning time μ is positive :

$$A_4 + \sqrt{(A_1A_3 - A_4)^2 - 4A_3(A_2 - A_1 - A_1A_4)} > A_1A_3$$

For this to hold we shall merely require that $A_2 - A_1 - A_1A_4 \leq 0$ (as A_3 is known to be positive). Assigning the values of A_1, A_2, A_4, this translates to $c_0 + c_0^2\frac{\gamma}{d} \leq O(c_0\gamma_1)$.

Dividing by c_0 we can now write $c_0 + \frac{c_0\sqrt{S_0}}{k} + k \leq dO(\gamma_1)$. As c_0 is the dominant element, we see that $d = \Omega\left(\frac{c_0}{\gamma_1}\right)$. which subsequently implies that : $\Omega(\sqrt{S_0}) \leq k \leq O(c_0)$. Therefore, we select the value of k such that $k = \Theta(\sqrt{S_0})$.

Theorem 5. *The time it takes a group of $k = \Theta(\sqrt{S_0})$ robots using the* ***SWEEP*** *algorithm to cover a connected region of size S_0 tiles, that expands every $d = \Omega(\frac{c_0}{\gamma_1})$ time steps (where γ_1 is defined in Result 3), is upper bounded by $O\left(S_0^2 \ln S_0\right)$.*

Proof. *Recalling Result 1, as we want at least a single contamination spread we assume $\frac{8(|\partial F_0|-1)\cdot(W(F_0)+k)}{k} + 2k \geq d$. Observing Result 3 we now see that :*

$$t_{SUCCESS} = d \cdot O\left(A_1 + \frac{|A_4|}{A_3} + \sqrt{A_1^2 + \frac{|A_1A_4| + A_1 + A_2}{A_3} + \frac{A_4^2}{A_3^2}}\right) \leq$$

$$d \cdot O(\, c_0 + \gamma_2 + dk\frac{\gamma_1}{\gamma_2} + k\gamma + \sqrt{k}\frac{\sqrt{c_0+\gamma_2}\sqrt{d\gamma_1+\gamma_2\cdot\gamma}}{\sqrt{\gamma_2}} + \sqrt{\frac{kd}{\gamma_2}}\sqrt{c_0+\gamma_2}\,)$$

Using the fact that $\gamma_2 = \Theta(c_0)$ (Lemma 1) we can rewrite this expression as :

$$d \cdot O(c_0 + dk\frac{\gamma_1}{c_0} + k\gamma + \sqrt{k}\sqrt{d\gamma_1 + c_0\gamma} + \sqrt{kd}) \qquad (3)$$

As $W(F_0) = O(\sqrt{S_0})$, we can now upper bound d as follows : $d = O(\frac{\sqrt{S_0}\cdot c_0}{k} + c_0 + k)$. Therefore, $|\gamma|$ can now be written as $|\gamma| = O(\frac{\sqrt{S_0}}{k} + \sqrt{S_0} + \frac{k}{\sqrt{S_0}} + 1)$. Remembering that $O(\sqrt{S_0}) \leq c_0 \leq O(S_0)$ we can rewrite Equation 3 as follows :

$$d \cdot O\left(c_0 + dk\frac{\gamma_1}{c_0} + k\sqrt{S_0} + \frac{k^2}{\sqrt{S_0}} + \sqrt{kc_0}\left(\sqrt{\gamma_1} + \sqrt[4]{S_0}\sqrt{\frac{\gamma_1}{k}} + \sqrt[4]{S_0} + \frac{\sqrt{k}}{\sqrt[4]{S_0}}\right) + \sqrt{kd}\right)$$

which 2 can simplify to $d \cdot O(c_0 + k\sqrt{S_0}\ln S_0 + \frac{k^2}{\sqrt{S_0}} + \sqrt{c_0 \ln S_0\sqrt{S_0}} + \sqrt{c_0 k\sqrt{S_0}})$.

Assuming that $k > O(\ln S_0)$ and as $c_0 = O(S_0)$ we can now write :

$$O\left(\frac{S_0^{2.5}}{k} + S_0^2\ln S_0 + S_0^{1.5}k\ln S_0 + k^2\sqrt{S_0}\ln S_0 + \frac{k^3}{\sqrt{S_0}} + \frac{S_0^{2.25}}{\sqrt{k}} + S_0^{1.75}\sqrt{k} + S_0^{0.75}k^{1.5}\right)$$

5 Conclusions

In this paper we have discussed the covering of a connected region on the grid using collaborate ant-like robotics system. We have shown that for static regions this can be done in $O(\frac{1}{k}\sqrt{n} \cdot c_0 + c_0 + k)$ time, equals $O(\frac{1}{k}n^{1.5} + n)$ time in the worst case. In addition, we have shown that when a region expands in a constant rate, a team of $\Theta(\sqrt{n})$ robots can still be guaranteed to clean or cover it, in $O(n^2 \ln n)$ time.

In addition, we have shown that teams of less than $\Omega(\sqrt{n})$ robots can *never* cover a region that expands every $O(1)$ time steps, regardless of their sensing capabilities, communications and memory resources, or algorithm used. As to regions that expand slower, two impossibility results were shown. First, a region of n tiles that expands every d time steps cannot be covered by a group of k agents if $dk \leq O(\sqrt{n})$. Theorem 4 guarantees a coverage when $dk = \Omega(\frac{n^{1.5}}{\ln n})$, or even for $dk = \Omega(n))$ (when the region's perimeter c_0 equals $O(n)$).

Second, a spreading region cannot be covered when d^2k is smaller than the order of the perimeter of the bounding rectangle of the region (which is $O(n)$ in the worse case and $O(\sqrt{n})$ for "round shapes"). Theorem 4 guarantees a coverage when $d^2k = \Omega(\frac{n^{2.5}}{\ln^2 n})$, or for $d^2k = \Omega(n^{1.5})$ (when $c_0 = O(n)$).

References

1. Altshuler, Y., Bruckstein, A., Wagner, I.: Swarm robotics for a dynamic cleaning problem. In: IEEE Swarm Intelligence Symposium. pp. 209–216 (2005)
2. Altshuler, Y., Wagner, I., Yanovski, V., Bruckstein, A.: Multi-agent cooperative cleaning of expanding domains. International Journal of Robotics Research (2010)
3. Altshuler, Y., Yanovsky, V., Bruckstein, A., Wagner, I.: Efficient cooperative search of smart targets using uav swarms. ROBOTICA 26, 551–557 (2008)
4. Borie, R., Tovey, C., Koenig, S.: Algorithms and complexity results for pursuit-evasion problems. The Intl. Joint Conf. on AI (IJCAI) pp. 59–66 (2009)
5. Dudek, G., Jenkin, M., Milios, E., Wilkes, D.: Robotic exploration as graph construction. IEEE Transactions on Robotics and Automation 7, 859–865 (1991)
6. Isler, V., Kannan, S., Khanna, S.: Randomized pursuit-evasion with local visibility. SIAM Journal of Discrete Mathematics 20, 26–41 (2006)
7. Koenig, S., Liu, Y.: Terrain coverage with ant robots: A simulation study. In: AGENTS'01
8. Koenig, S., Szymanski, B., Liu, Y.: Efficient and inefficient ant coverage methods. Annals of Mathematics and Artificial Intelligence 31, 41–76 (2001)
9. Szymanski, B., Koenig, S.: The complexity of node counting on undirected graphs. Technical Report, CS Department, Rensselaer Polytechnic Institute (1998)
10. Thrun., S.B.: Efficient exploration in reinforcement learning — technical report cmu-cs-92-102. Technical report, Carnegie Mellon University (1992)
11. Wagner, I., Altshuler, Y., Yanovski, V., Bruckstein, A.: Cooperative cleaners: A study in ant robotics. The Intl. J. of Robotics Res. (IJRR) 27(1), 127–151 (2008)
12. Wagner, I., Lindenbaum, M., Bruckstein, A.: Efficiently searching a graph by a smell-oriented vertex process. Annals of Math. and AI 24, 211–223 (1998)