AIMS CDT - Signal Processing

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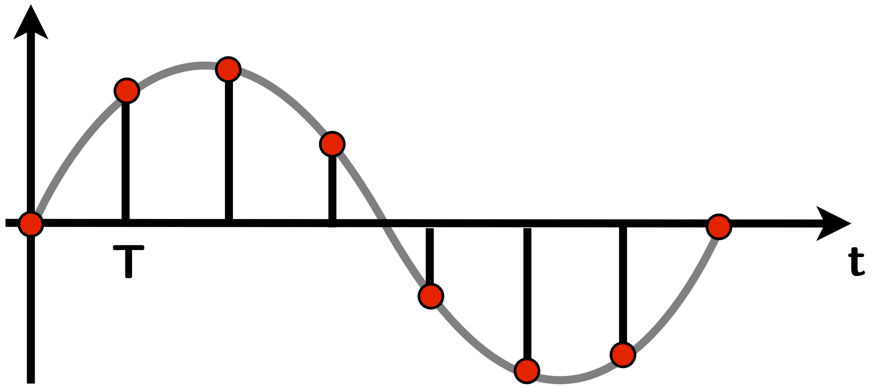
Representation of Signals

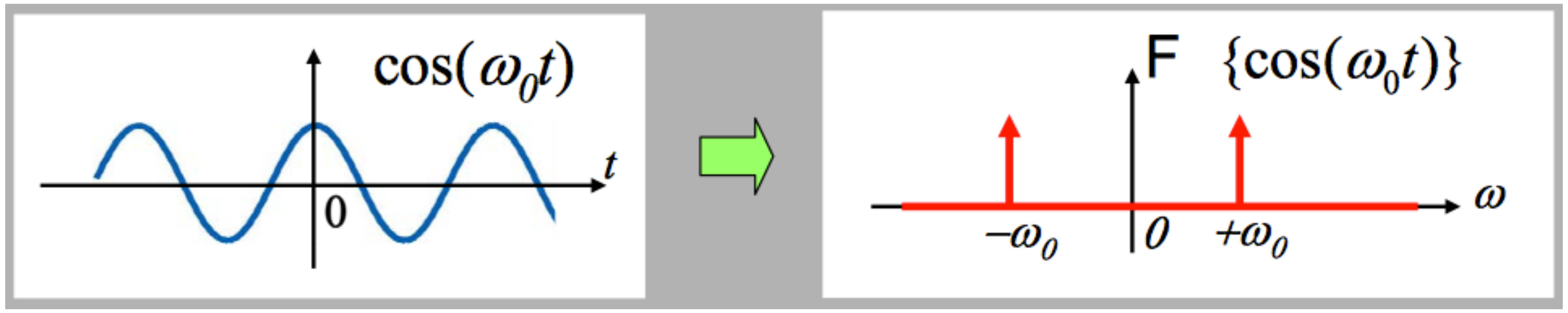
What is a representation of a signal?

- Sum of delta functions in time or space (sampling domain)
  - good for display or playback
  - not good for analysis (e.g., denoising, compression)



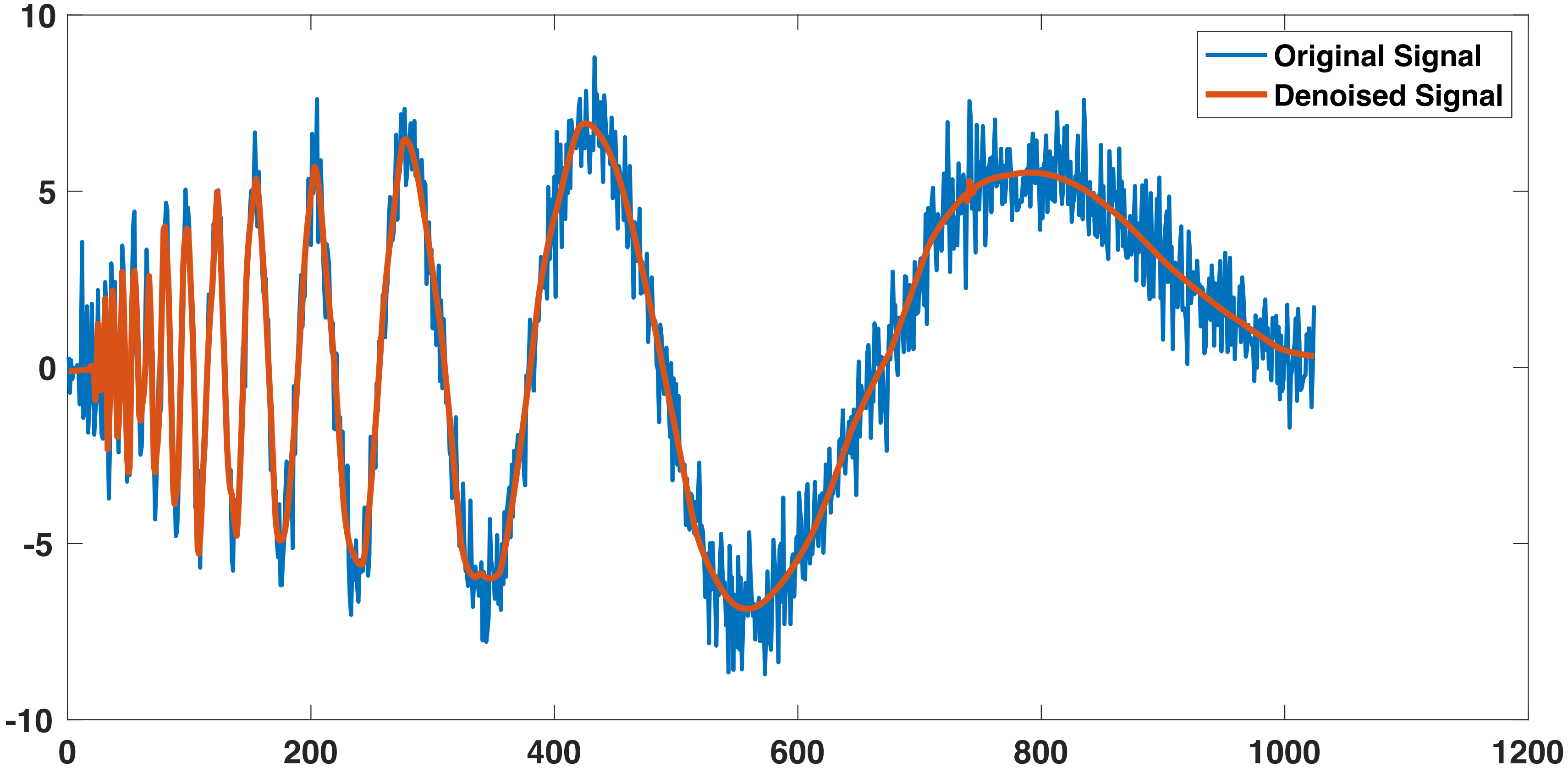
What is a representation of a signal?

- In a time-series setting, useful representation could be past samples
- More generally, it involves transformation of the signal into a new domain
  - where signal characteristics are revealed
  - example: **Fourier coefficients** reveal **rate of change** of the signal



- Usefulness of the representation depends on the analysis goal
  - which may vary but all share the core desire for **simplification**

Example: Denoising



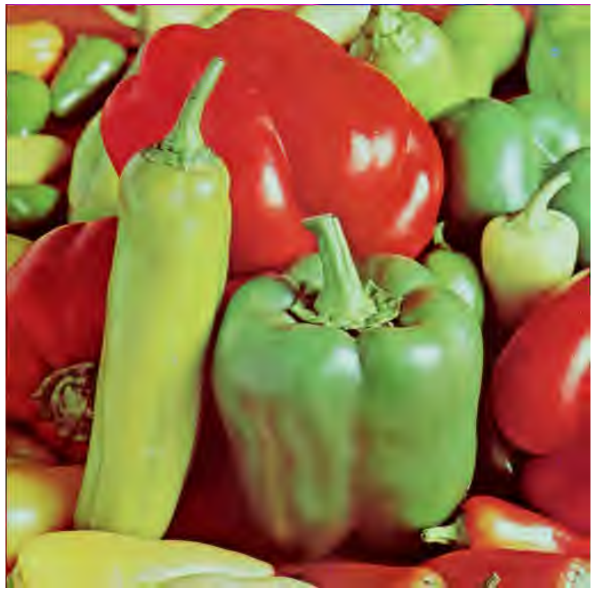
goal: recover signal from noisy observation

Example: Compression

original

JPEG 2000 (10% in size)

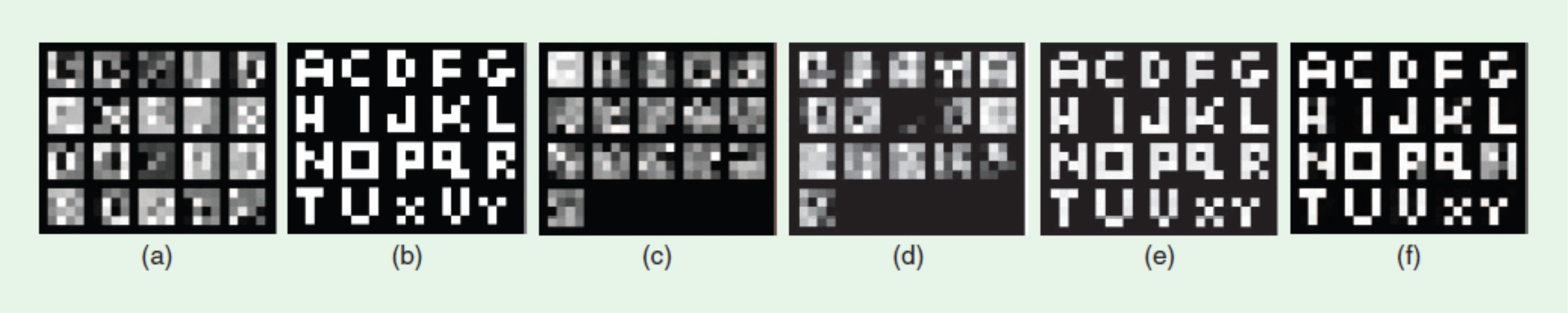
JPEG 2000 (1% in size)



goal: compress file size without sacrificing quality

Example: Recognition

samples **true “causes”** **PCA** **ICA** **sparse coding** **KSVD**



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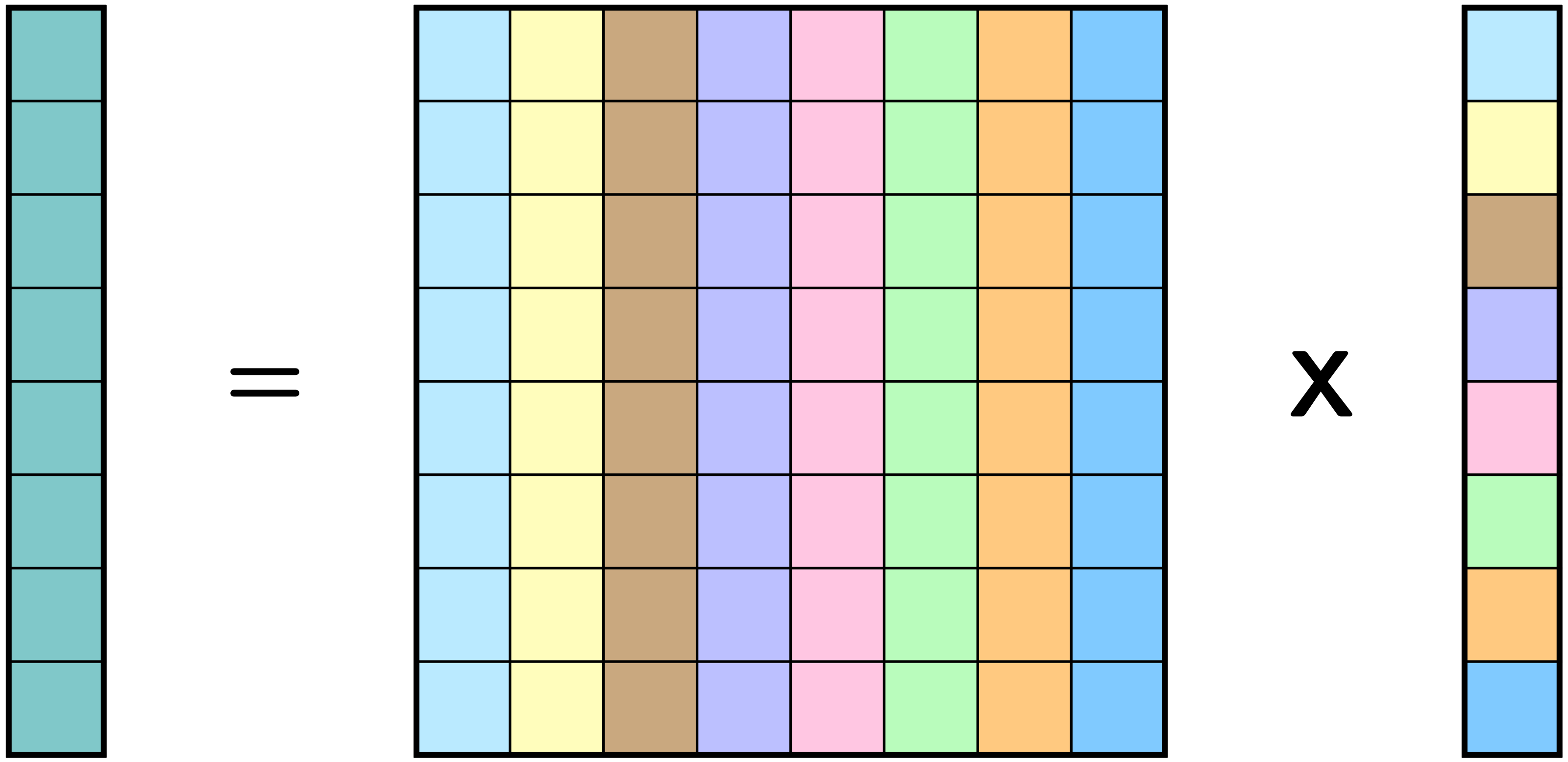
goal: capture true “causes” of signals

Signal representation via dictionaries



signal **dictionary** **coefficients**

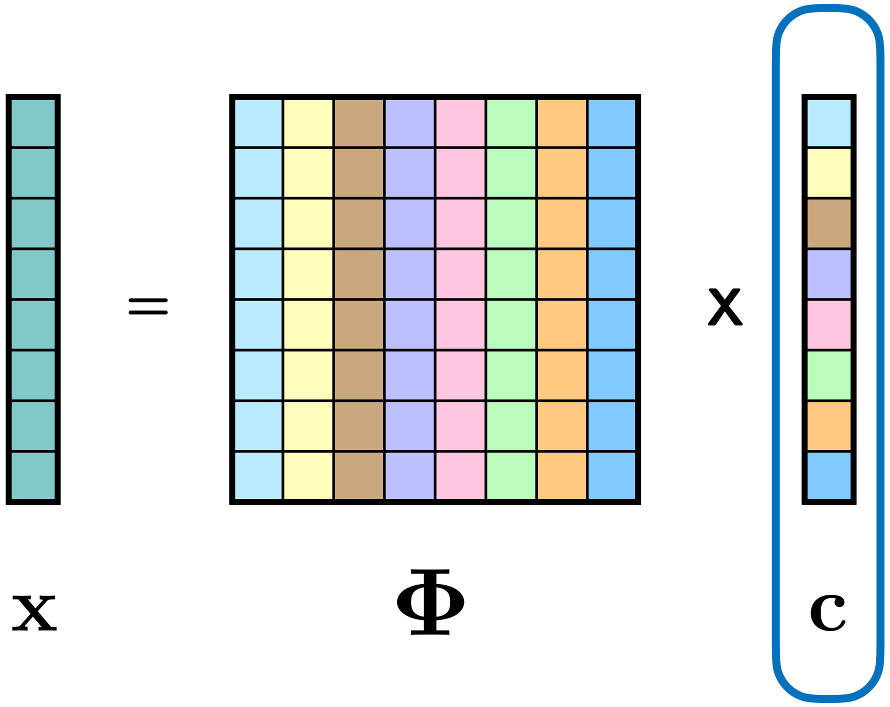
$\mathbf{x}$ $\quad \boldsymbol{\Phi} = [\boldsymbol{\phi}_{\mathbf{0}}\ \boldsymbol{\phi}_{\mathbf{1}}\ \cdots\ \boldsymbol{\phi}_{\boldsymbol{N-1}}]$ $\quad \mathbf{c}$

“atoms”

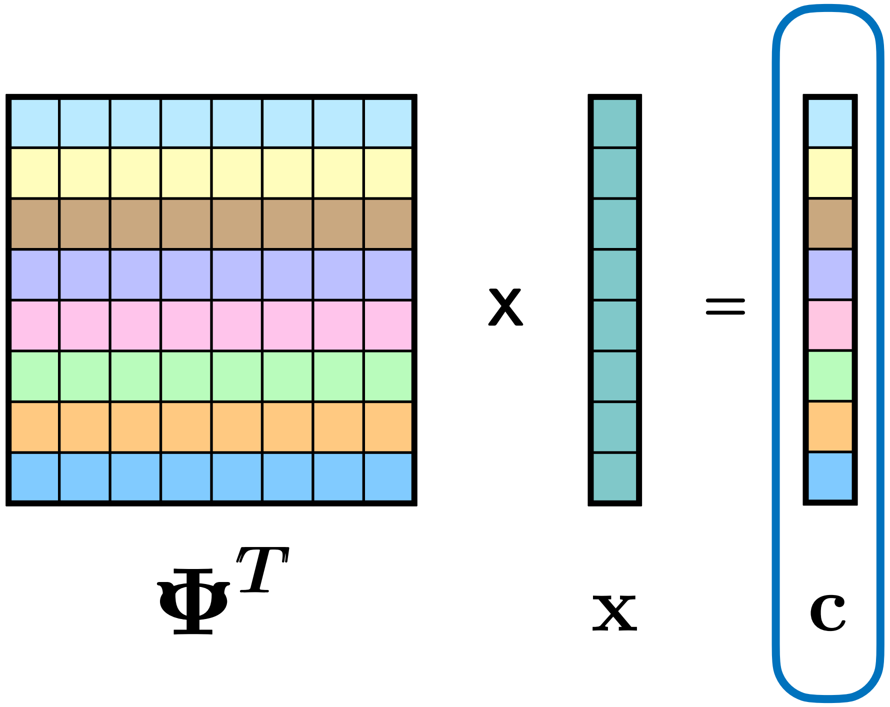
Signal representation via dictionaries

- Complete dictionaries

synthesis



analysis



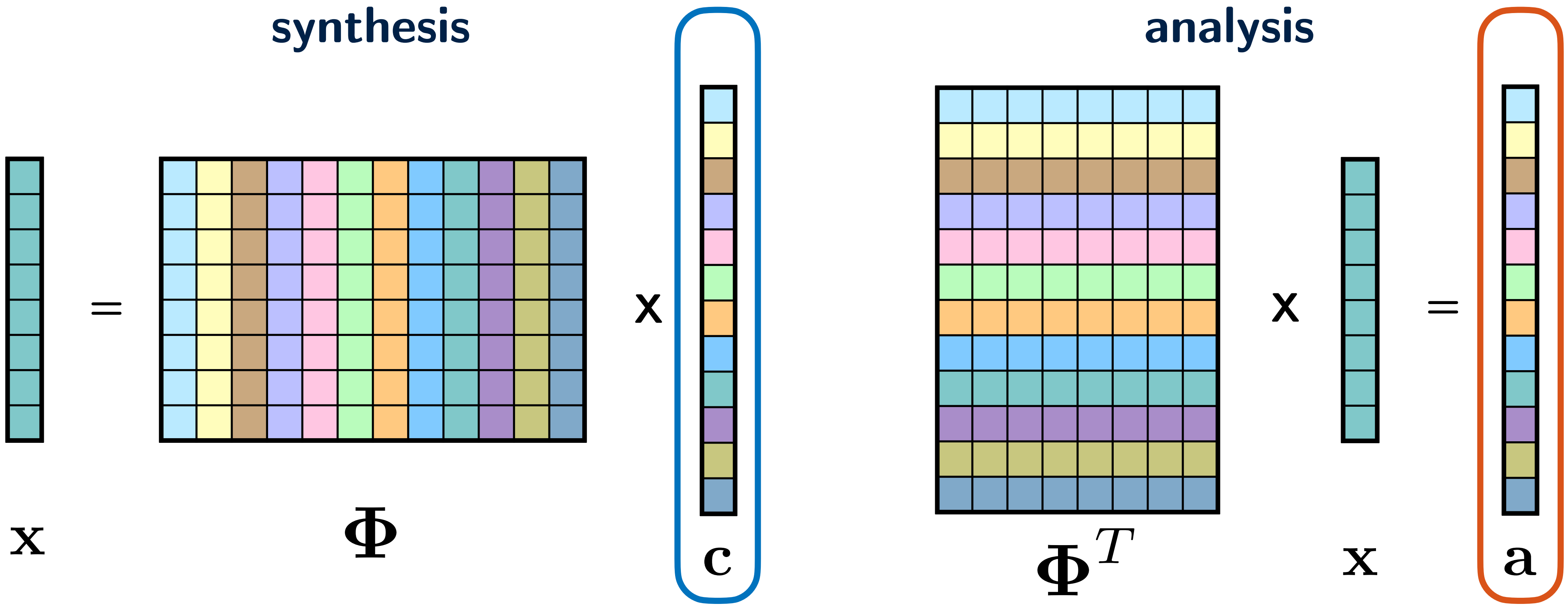
$$\mathbf{x} = \boldsymbol{\Phi}\mathbf{c} = \sum_n \mathbf{c}_n \phi_n$$

$$\mathbf{c}_n = \boldsymbol{\phi}_n^T \mathbf{x}$$

equivalent for complete dictionaries

Signal representation via dictionaries

- Overcomplete dictionaries

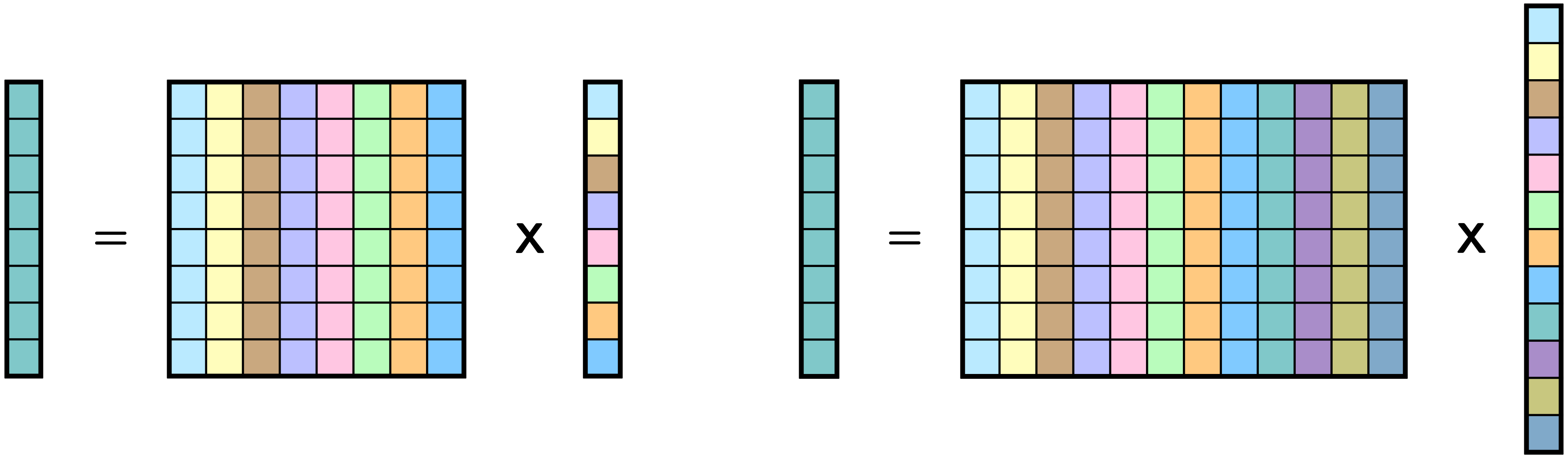


$$\mathbf{x} = \mathbf{\Phi}\mathbf{c} = \sum_n \mathbf{c}_n \phi_n$$

$$\mathbf{a}_n = \phi_n^T \mathbf{x}$$

not equivalent for over-complete dictionaries

Signal representation via dictionaries



design of dictionaries

- from mathematical modelling of data (transforms/analytic dictionaries)
- from a set of realisations of data (dictionary learning)

Outline

- A historical overview of dictionary design techniques
  - signal representation via stochastic models
  - transforms & analytic dictionaries
  - trained dictionaries (dictionary learning)
- Discussion
  - applications
  - connection with deep learning

1920s-30s: Stochastic models

- Stochastic models
  - examples of parametric models
  - describe how data were generated
  - provide a special representations of signal from a time-series viewpoint
- Typical examples
  - autoregressive (AR) models
  - moving average (MA) models
  - autoregressive moving average (ARMA) models

Autocorrelation

- Autocovariance (covariance between the signal and a lagged version of itself)

$$\sigma_{xx}(T) = \frac{1}{N-1} \sum_{t=1}^{N} (\underbrace{x_{t-T}}_{\text{lagged version by T samples}} - \mu_x)(\underbrace{x_t}_{\text{signal}} - \mu_x)$$

- Autocorrelation (normalised autocovariance)

$$r_{xx}(T) = \frac{\sigma_{xx}(T)}{\sigma_{xx}(0)} \qquad \text{where} \quad \sigma_{xx}(0) = \sigma_x^2$$

- Both are **symmetric** or **even** functions

Autocorrelation

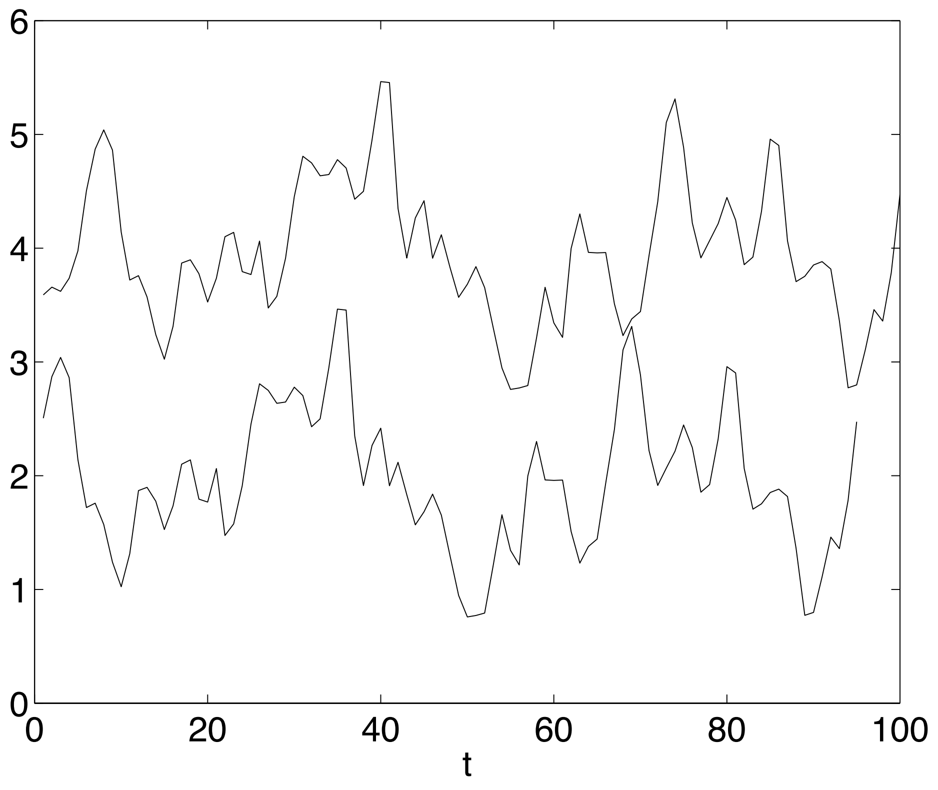


Figure 4.7: *Signal x_t (top) and x_{t+5} (bottom). The bottom trace* **leads** *the top trace by 5 samples. Or we may say it* **lags** *the top by -5 samples.*

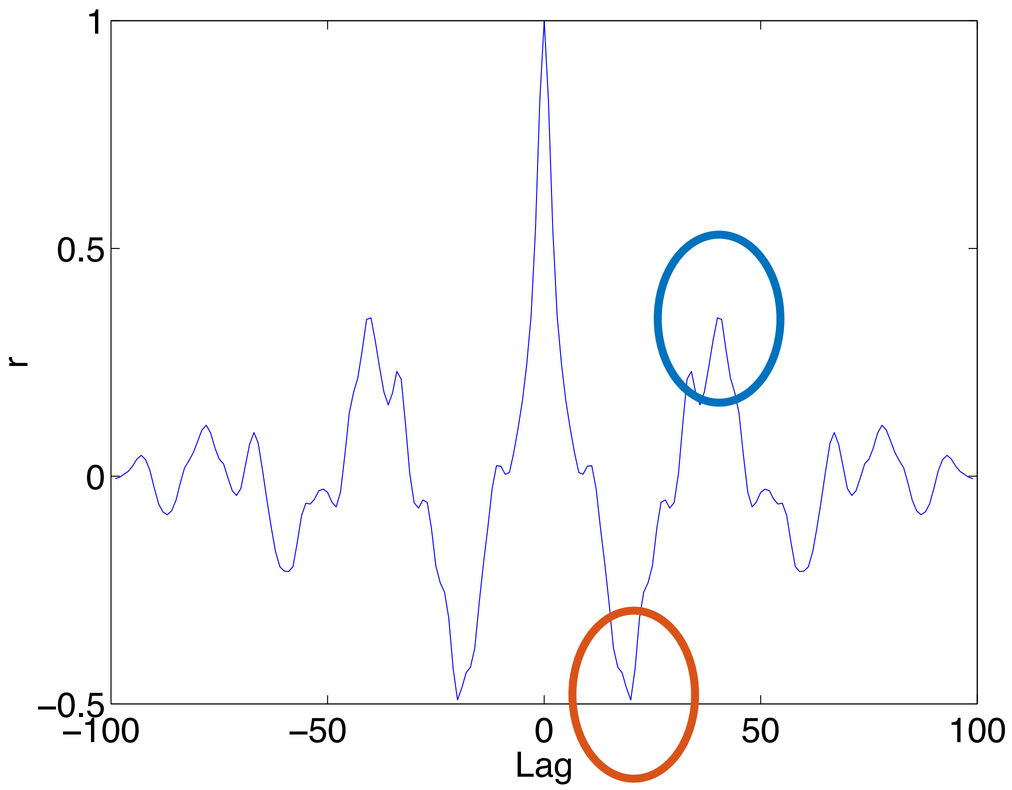


Figure 4.8: *Autocorrelation function for x_t. Notice the negative correlation at lag 20 and positive correlation at lag 40. Can you see from Figure 4.7 why these should occur?*

Autoregressive (AR) models

- An AR model **predicts** the value of a time-series from previous values

$$x_t = \sum_{i=1}^{p} x_{t-i} \boxed{a_i} + \boxed{e_t}$$

AR coefficients **prediction error** $e_t \sim \mathcal{N}(0, \sigma_e^2)$

- Matrix form

$$\mathbf{M} = \begin{bmatrix} x_4 & x_3 & x_2 & x_1 \\ x_5 & x_4 & x_3 & x_2 \\ .. & .. & .. & .. \\ x_{N-1} & x_{N-2} & x_{N-3} & x_{N-4} \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_5 \\ x_6 \\ .. \\ x_N \end{bmatrix} = \begin{bmatrix} x_4 & x_3 & x_2 & x_1 \\ x_5 & x_4 & x_3 & x_2 \\ .. & .. & .. & .. \\ x_{N-1} & x_{N-2} & x_{N-3} & x_{N-4} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} + \begin{bmatrix} e_5 \\ e_6 \\ .. \\ e_N \end{bmatrix}$$

embedding matrix

$$\mathbf{x} = \mathbf{M}\mathbf{a} + \mathbf{e}$$

Autoregressive (AR) models

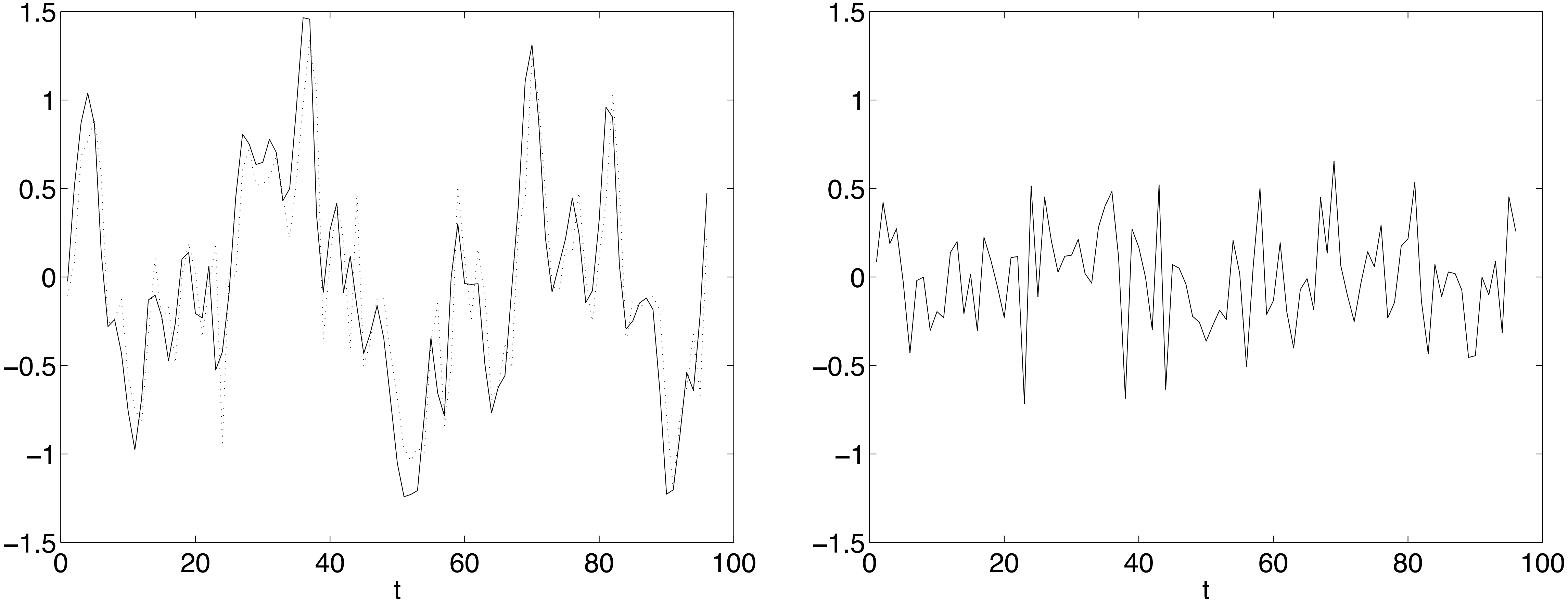
- The AR model is a special case of the multivariate regression model
- It also provides a special **representation** of the signal
- To compute AR coefficients and predictions

$$\mathbf{x} = \mathbf{M}\mathbf{a} + \mathbf{e} \quad \Longrightarrow \quad \begin{aligned} \hat{\mathbf{a}} &= (\mathbf{M}^T\mathbf{M})^{-1}\mathbf{M}^T\mathbf{x} \\ \hat{\mathbf{x}} &= \mathbf{M}\hat{\mathbf{a}} \\ \mathbf{e} &= \mathbf{x} - \hat{\mathbf{x}} \end{aligned}$$

Autoregressive (AR) models

- Use an AR(4) model to analyse data shown before:

$$\hat{\mathbf{a}} = [1.46, -1.08, 0.60, -0.186]^T \qquad \sigma_e^2 = 0.079 \qquad \sigma_x^2 = 0.3882$$



(a)

(b)

Figure 4.9: *(a) Original signal (solid line),* $\mathbf{x}$*, and predictions (dotted line),* $\hat{\mathbf{x}}$*, from an AR(4) model and (b) the prediction errors,* $\mathbf{e}$*. Notice that the variance of the errors is much less than that of the original signal.*

Autoregressive (AR) models

- Relation to autocorrelation

$$x_t = a_1 x_{t-1} + a_2 x_{t-2} + \ldots + a_p x_{t-p} + e_t$$

⬇ multiply by x_{t-k}

$$x_t x_{t-k} = a_1 x_{t-1} x_{t-k} + a_2 x_{t-2} x_{t-k} + \ldots + a_p x_{t-p} x_{t-k} + e_t x_{t-k}$$

⬇ sum over t and divide by N-1

$$\sigma_{xx}(k) = a_1 \sigma_{xx}(k-1) + a_2 \sigma_{xx}(k-2) + \ldots + a_p \sigma_{xx}(k-p) + \sigma_{e,x}$$

⬇ divide by signal variance

$$r_{xx}(k) = a_1 r_{xx}(k-1) + a_2 r_{xx}(k-2) + \ldots + a_p r_{xx}(k-p)$$

Autoregressive (AR) models

- For an AR(4) model

$$\begin{bmatrix} r_{xx}(1) \\ r_{xx}(2) \\ r_{xx}(3) \\ r_{xx}(4) \end{bmatrix} = \begin{bmatrix} r_{xx}(0) & r_{xx}(-1) & r_{xx}(-2) & r_{xx}(-3) \\ r_{xx}(1) & r_{xx}(0) & r_{xx}(-1) & r_{xx}(-2) \\ r_{xx}(2) & r_{xx}(1) & r_{xx}(0) & r_{xx}(-1) \\ r_{xx}(3) & r_{xx}(2) & r_{xx}(1) & r_{xx}(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

Yule-Walker relations

$$\mathbf{r} = \mathbf{R}\mathbf{a}$$

- More efficient way to estimate AR coefficients: $\mathbf{a} = \mathbf{R}^{-1}\mathbf{r}$

$$\mathbf{R} = \begin{bmatrix} 1 & r_{xx}(1) & r_{xx}(2) & r_{xx}(3) \\ r_{xx}(1) & 1 & r_{xx}(1) & r_{xx}(2) \\ r_{xx}(2) & r_{xx}(1) & 1 & r_{xx}(1) \\ r_{xx}(3) & r_{xx}(2) & r_{xx}(1) & 1 \end{bmatrix}$$

- autocorrelation matrix is symmetric and Toeplitz
- efficient computation via a recursive estimation technique (Levinson-Durbin)

Multivariate processes

- Cross-covariance (covariance between one signal and the lagged version of another)

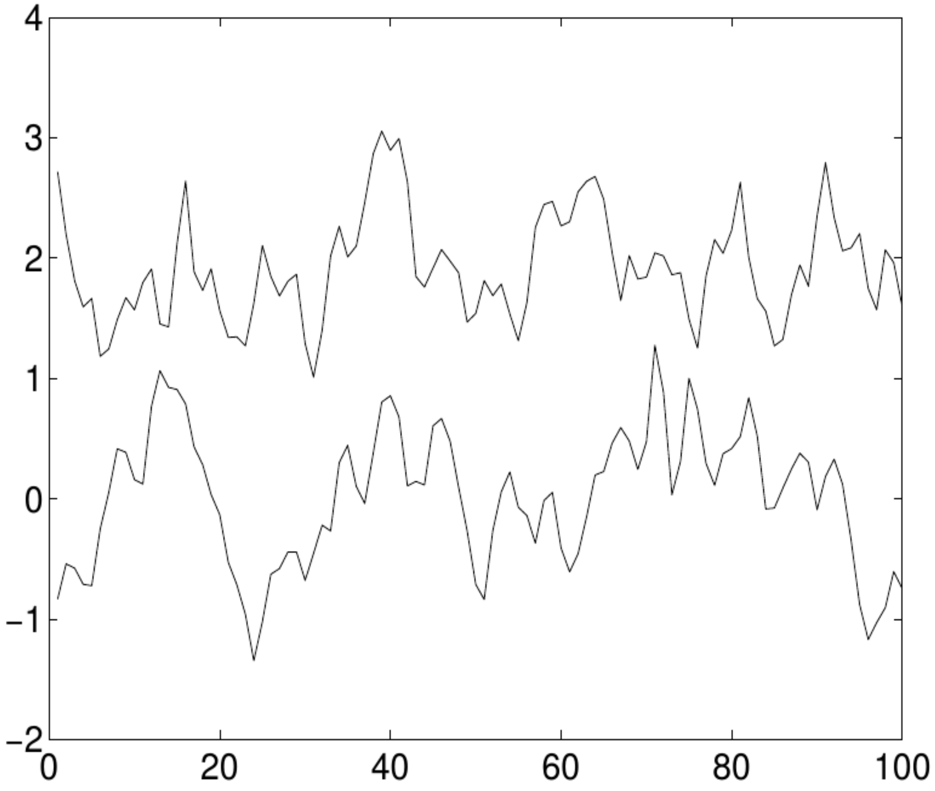
$$\sigma_{xy}(T) = \frac{1}{N-1} \sum_{t=1}^{N} (x_{t-T} - \mu_x)(y_t - \mu_y)$$

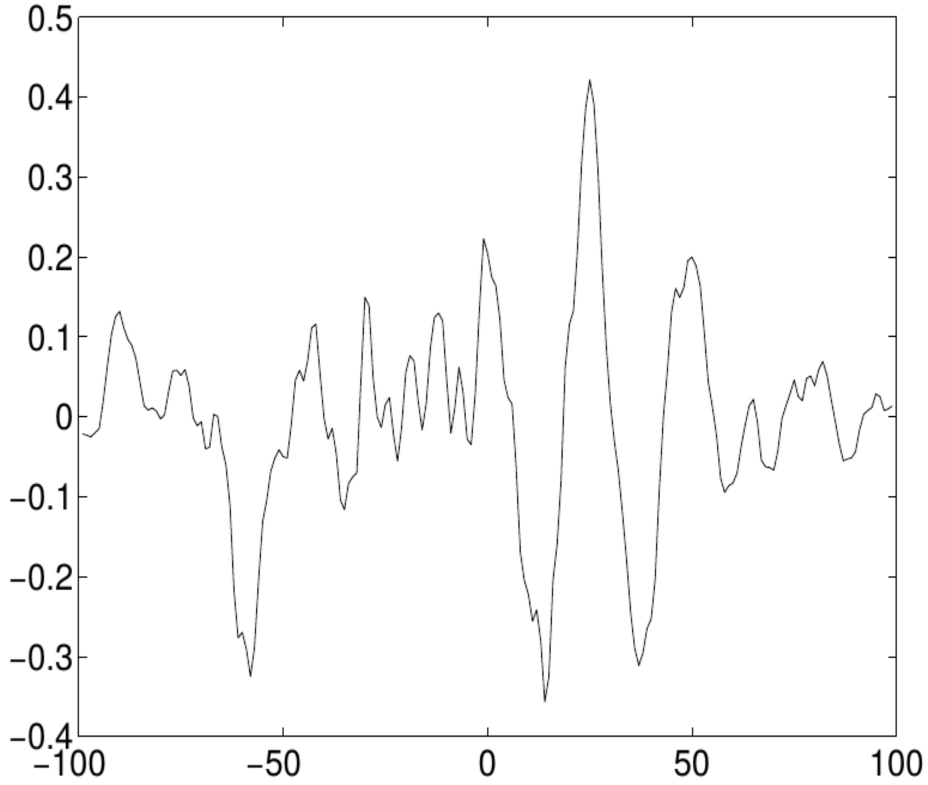
- Cross-correlation

$$r_{xy}(T) = \frac{\sigma_{xy}(T)}{\sqrt{\sigma_{xx}(0)\sigma_{yy}(0)}}$$

- This is **not symmetric** and **has non-unity at lag 0** (unlike autocorrelation)

Cross-correlation





Multivariate AR (MAR) models

- Univariate

$$x_t = \sum_{i=1}^{p} x_{t-i} a_i + e_t$$

- Multivariate

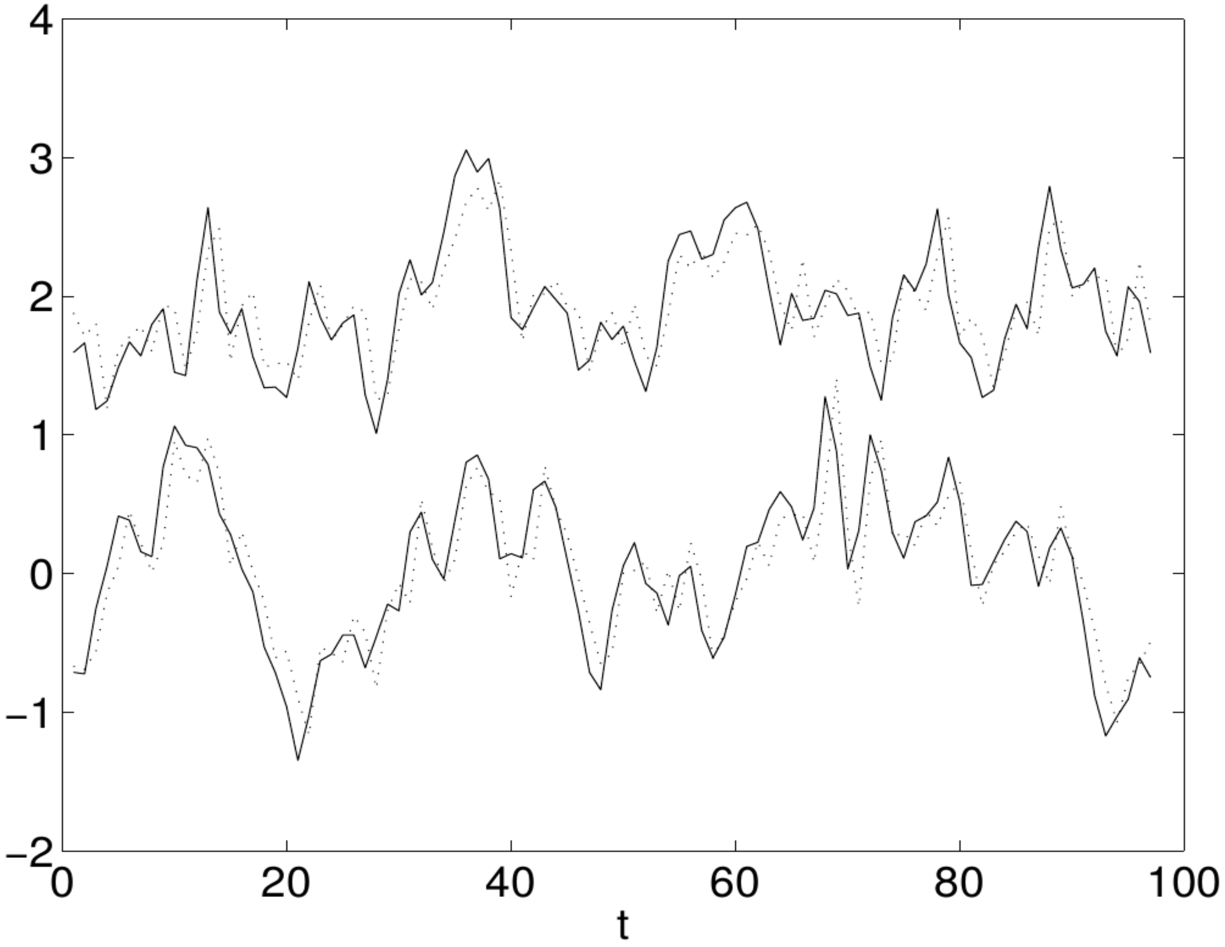
for time t:

$$\underbrace{\left[\, x_t^{(1)},\ x_t^{(2)} \,\right]}_{\mathbf{x}_t} = \underbrace{\left[\, x_{t-1}^{(1)}, x_{t-1}^{(2)}, x_{t-2}^{(1)}, x_{t-2}^{(2)}, \cdots, x_{t-p}^{(1)}, x_{t-p}^{(2)} \,\right]}_{\tilde{\mathbf{x}}_t} \underbrace{\begin{bmatrix} a_1^{(11)}, & a_1^{(12)} \\ a_1^{(21)}, & a_1^{(22)} \\ a_2^{(11)}, & a_2^{(12)} \\ a_2^{(21)}, & a_2^{(22)} \\ \cdots, & \cdots \\ a_p^{(11)}, & a_p^{(12)} \\ a_p^{(21)}, & a_p^{(22)} \end{bmatrix}}_{\mathbf{A}} + \underbrace{\left[\, e_t^{(1)},\ e_t^{(2)} \,\right]}_{\mathbf{e}_t}$$

- Extending to multiple time steps (stacking $\tilde{\mathbf{x}}_t$ into embedding matrix $\tilde{\mathbf{M}}$)

$$\mathbf{X} = \tilde{\mathbf{M}}\mathbf{A} + \mathbf{E} \quad \Rightarrow \quad \hat{\mathbf{A}} = (\tilde{\mathbf{M}}^T \tilde{\mathbf{M}})^{-1} \tilde{\mathbf{M}}^T \mathbf{X}$$

Multivariate AR (MAR) models



solid: original multivariate time-series

dashed: predictions

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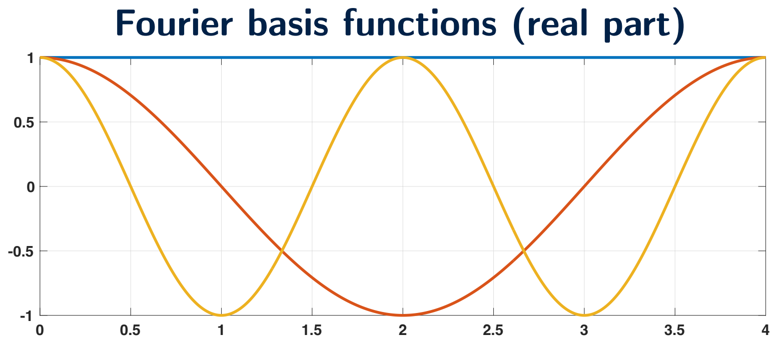
1960s: Fourier basis and DFT

- recall the LTI system

$$x(t) \quad \bigcirc \longrightarrow \boxed{g(t)} \longrightarrow y(t)$$



$$y(t) = \int_{-\infty}^{\infty} e^{j\omega(t-\tau)} g(\tau) d\tau = \boxed{e^{j\omega t}} G(\omega)$$

Fourier basis diagonalises convolution operator

$$X(\omega) \longrightarrow \boxed{G(\omega)} \longrightarrow Y(\omega) = X(\omega)G(\omega)$$

1960s: Fourier basis and DFT

discrete Fourier transform

1960s

- Fourier basis describes a signal in terms of its **global** frequency content and hence is good at representing **uniformly smooth** signals
- discrete Fourier transform (DFT) provides an **orthogonal** dictionary: $\boxed{\phi_n(k) = e^{j\frac{2\pi}{N}nk}}$

$$\begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{pmatrix} = \frac{1}{N} \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & W & W^2 & W^3 & \dots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \dots & W^{N-2} \\ 1 & W^3 & W^6 & W^9 & \dots & W^{N-3} \\ \vdots & & & & & \\ 1 & W^{N-1} & W^{N-2} & W^{N-3} & \dots & W \end{pmatrix} \begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{pmatrix} \quad \text{with} \quad W = e^{j\frac{2\pi}{N}}$$

- fast Fourier transform (FFT) reduces complexity from $\mathcal{O}(N^2)$ to $\mathcal{O}(N\log N)$

1960s: Fourier basis and DFT

discrete Fourier transform

1960s

- DFT produces complex coefficients (“wasteful” for real signals)
- DFT assumes periodic extension (discontinuity at boundary)

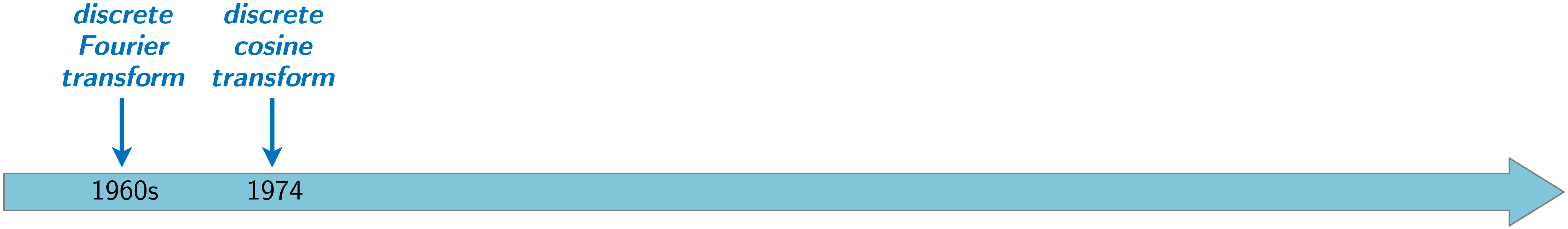
Fourier transform $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)[\cos(\omega t) - j\sin(\omega t)]dt$

$$\xrightarrow{x(t) = x(-t)} \quad X(\omega) = \int_{-\infty}^{\infty} x(t)\cos(\omega t)dt$$

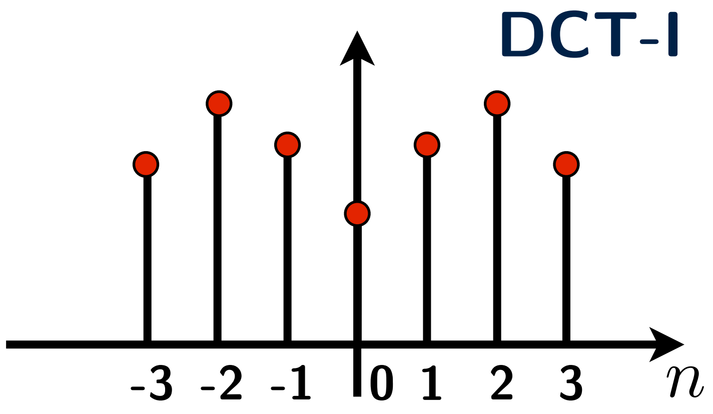
cosine transform

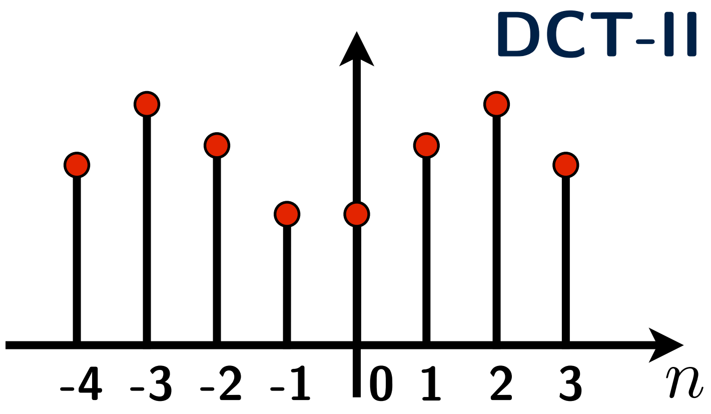
- a real and even signal leads to a **real** cosine transform

1970s: DCT



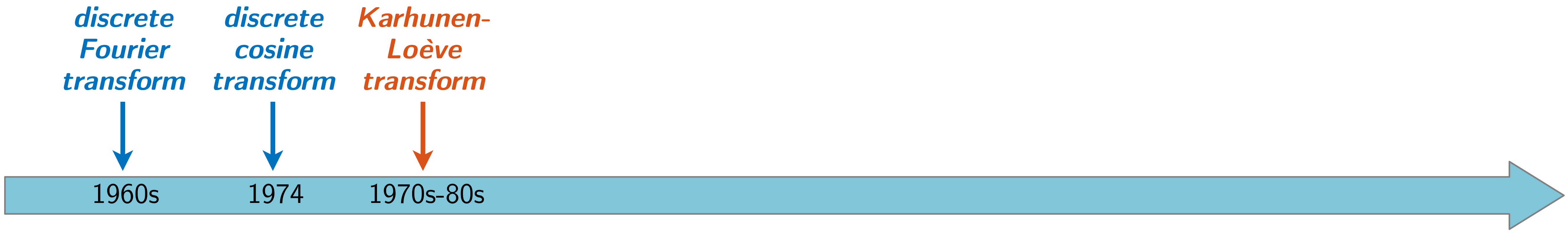
- the discrete version is called the discrete cosine transform (DCT)
- several variants of symmetric extension, which all make the signal **even** and lead to **smoother boundary**





- DCT-II provides a real dictionary: $\phi_n(k) = \cos[\frac{\pi}{N}(n + \frac{1}{2})k]$
- DCT-II is behind the JEPG image compression standard

1970s-80s: KLT and PCA



- projection onto a fixed subset of DFT or DCT atoms leads to **compaction**

$$\mathbf{x} \approx \sum_{n \in \mathcal{S}_k} (\boldsymbol{\Psi}_n^T \mathbf{x}) \boldsymbol{\Phi}_n$$

- but **data** themselves can also be a source of compaction
- Karhunen-Loève transform (KLT) or principal component analysis (PCA) fits a low-dimensional subspace to data

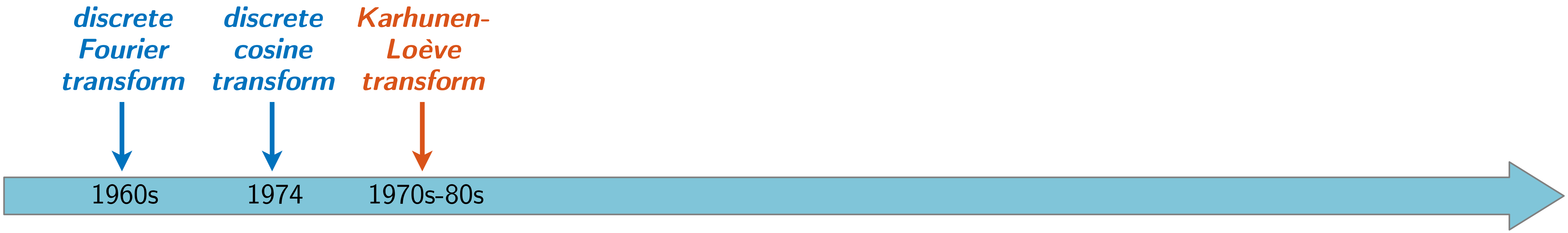
$$\boldsymbol{\Sigma} = \boldsymbol{\Phi} \boldsymbol{\Lambda} \boldsymbol{\Phi}^T$$

known/empirical covariance

eigenvectors (dictionary atoms as k largest eigenvectors)

- representation is efficient (maximally compacts energy) but expensive to compute

DCT vs KLT



discrete Fourier transform

1960s

discrete cosine transform

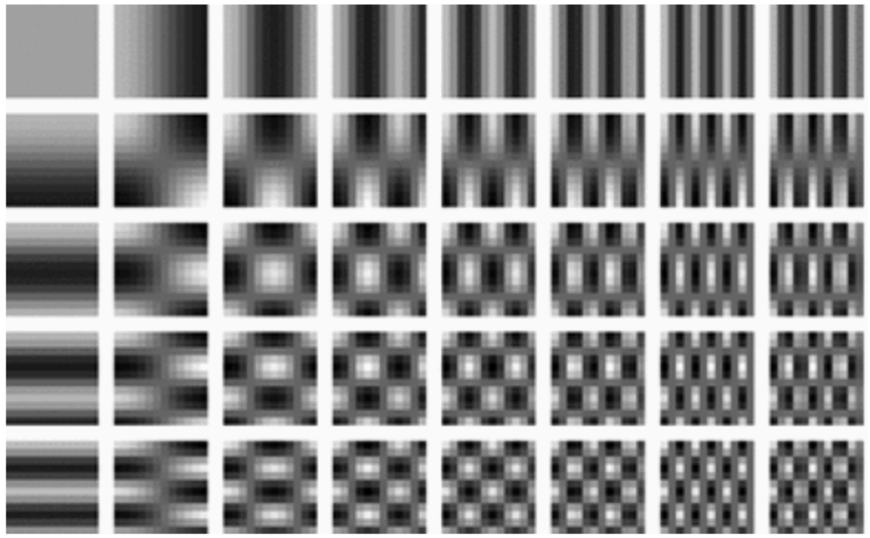
1974

Karhunen-Loève transform

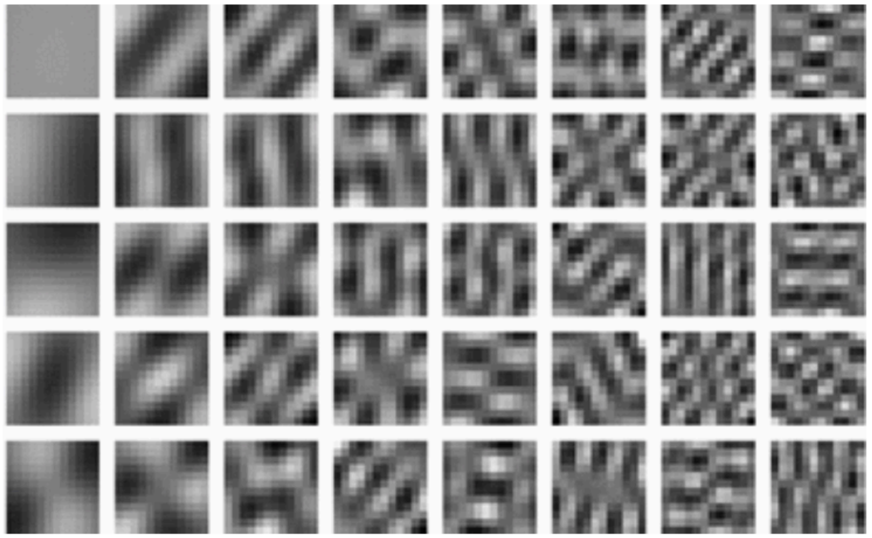
1970s-80s

- DCT atoms (12x12) vs. KLT atoms (trained using 12x12 image patches)

DCT

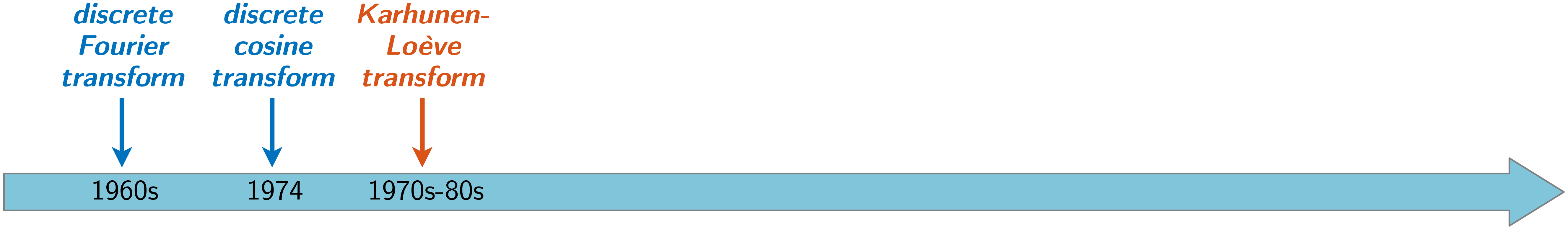


KLT



Rubinstein et al., "Dictionaries for sparse representation modeling," Proceedings of the IEEE, 2010.

The need for sparsity


discrete Fourier transform
discrete cosine transform
Karhunen-Loève transform
1960s
1974
1970s-80s

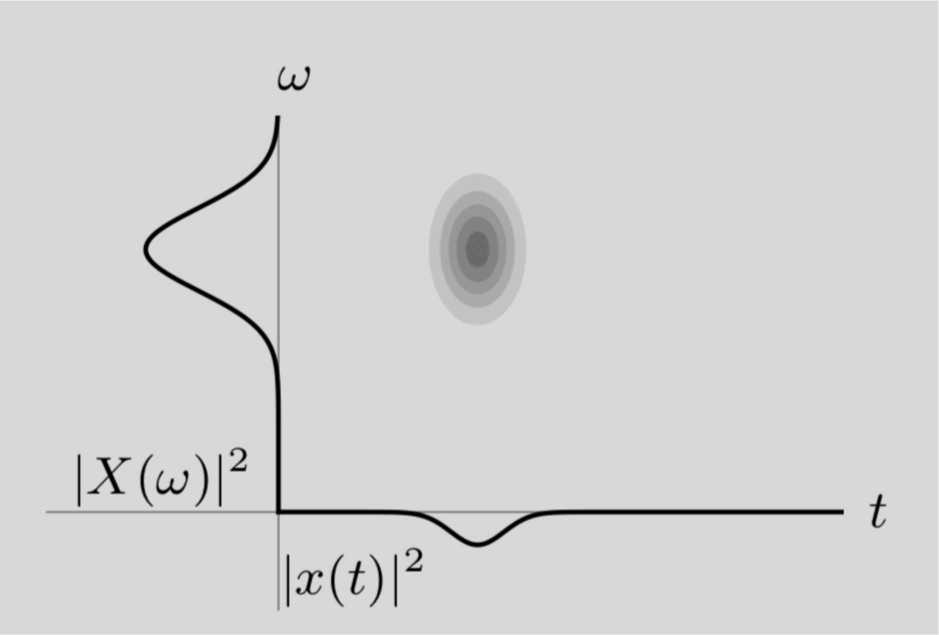
- simplicity motivates **sparsity**: signal as linear combination of a few atoms
- sparsity requires shift from linear to **nonlinear** approximation

$$\mathbf{x} \approx \sum_{\mathcal{K}} c_k \phi_k \longrightarrow$$

subset of atoms which is different for each x

- sparsity requires **localisation:** atoms with concentrated supports
  - allow more flexible representations based on local characteristics
  - limit effects of irregularities (a main source of large coefficients)

Time-frequency representation



time-frequency tile (Heisenberg box)

time-frequency plane

time localisation

$$\mu_t = \frac{1}{||x||^2} \int_{-\infty}^{\infty} t|x(t)|^2 dt$$

$$\Delta_t = \left(\frac{1}{||x||^2} \int_{-\infty}^{\infty} (t - \mu_t)^2 |x(t)|^2 dt \right)^{\frac{1}{2}}$$

frequency localisation

$$\mu_f = \frac{1}{2\pi||x||^2} \int_{-\infty}^{\infty} \omega|X(\omega)|^2 d\omega$$

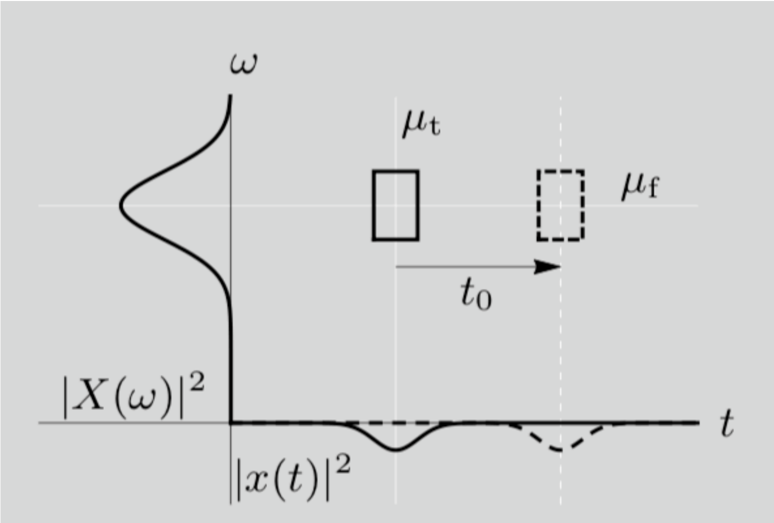
$$\Delta_f = \left(\frac{1}{2\pi||x||^2} \int_{-\infty}^{\infty} (\omega - \mu_f)^2 |X(\omega)|^2 d\omega \right)^{\frac{1}{2}}$$

Vetterli et al., "Foundations of signal processing," Cambridge University Press, 2014.

Time-frequency representation

- Consider three basic operations

shift in time

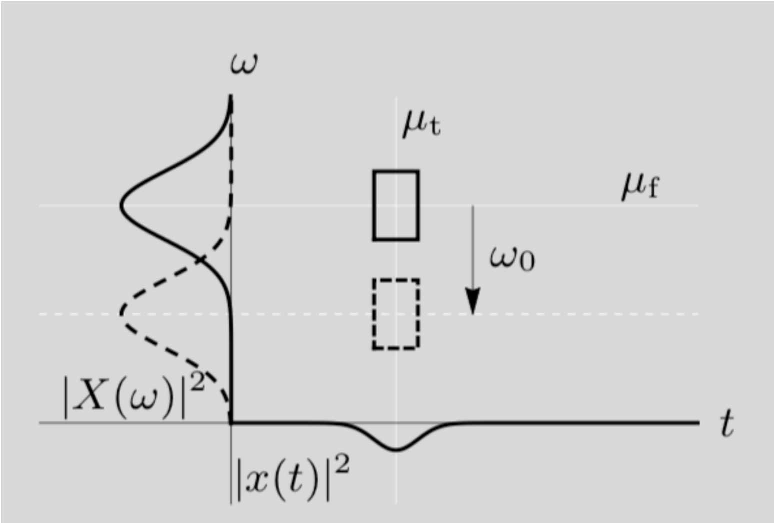


$$y(t) = x(t - t_0)$$

FT ⬇

$$Y(\omega) = e^{-j\omega t_0} X(\omega)$$

shift in frequency

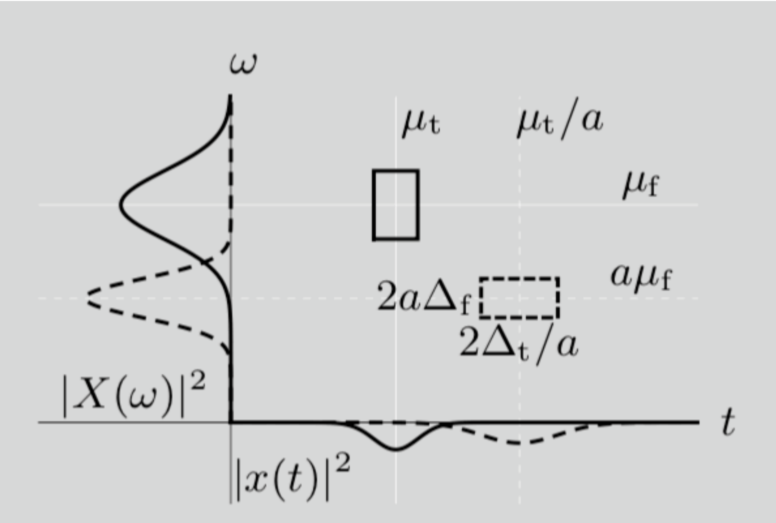


$$y(t) = e^{j\omega_0 t} x(t)$$

FT ⬇

$$Y(\omega) = X(\omega - \omega_0)$$

scaling in time



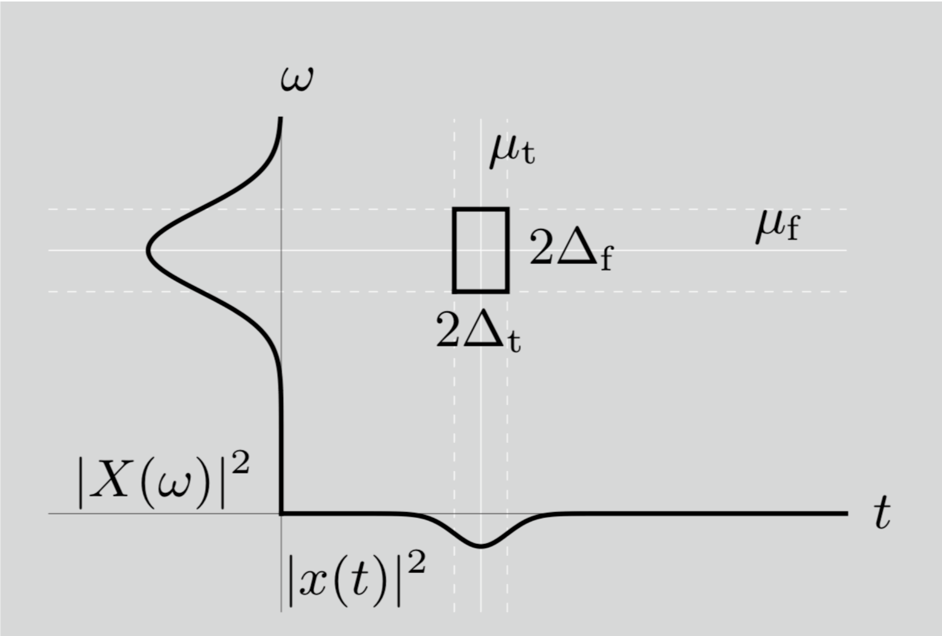
$$y(t) = \sqrt{a} x(at)$$

FT ⬇

$$Y(\omega) = \frac{1}{\sqrt{a}} X(\frac{\omega}{a})$$

Vetterli et al., "Foundations of signal processing," Cambridge University Press, 2014.

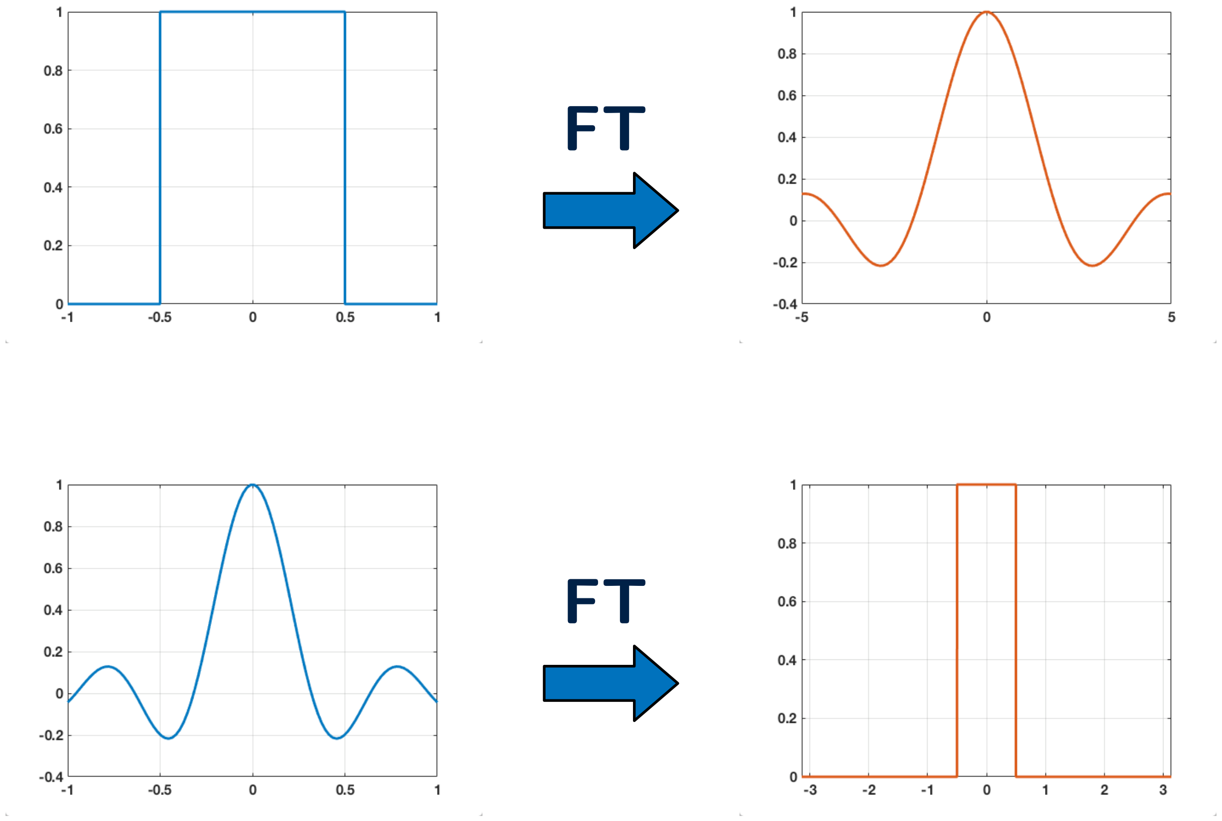
Time-frequency representation



Heisenberg's uncertainty principle

$$\text{Let } x \in \mathcal{L}^2(\mathbb{R}), \text{ then } \Delta_t \Delta_f \geq \frac{1}{2}$$

examples

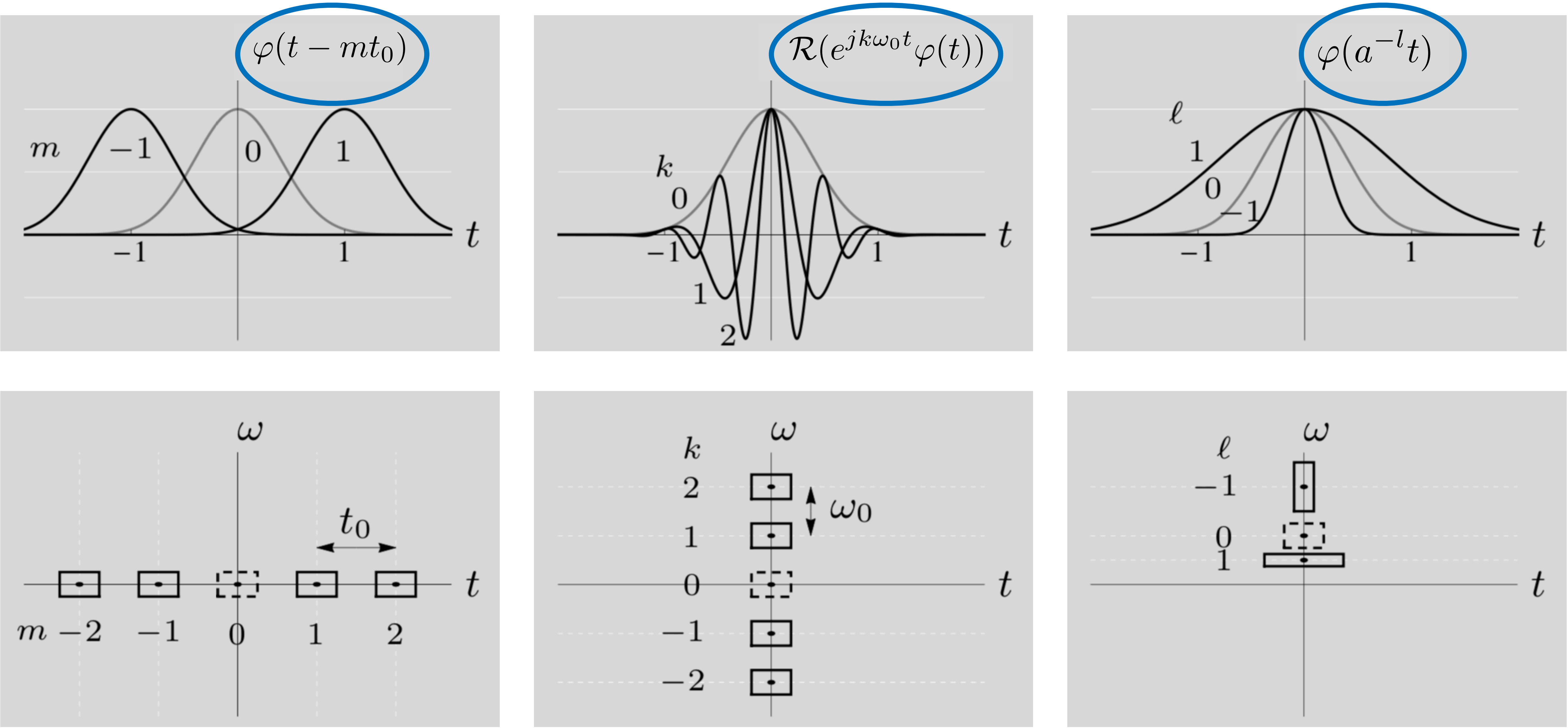


Vetterli et al., "Foundations of signal processing," Cambridge University Press, 2014.

Time-frequency representation

- Consider three structured sets of functions

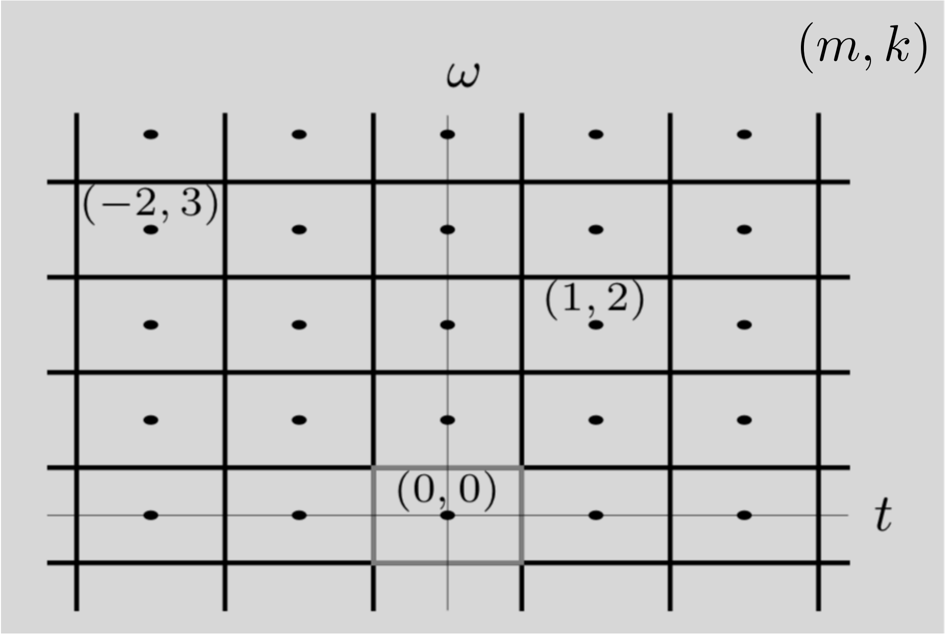
shift in time **shift in frequency** **scaling in time**



Vetterli et al., "Foundations of signal processing," Cambridge University Press, 2014.

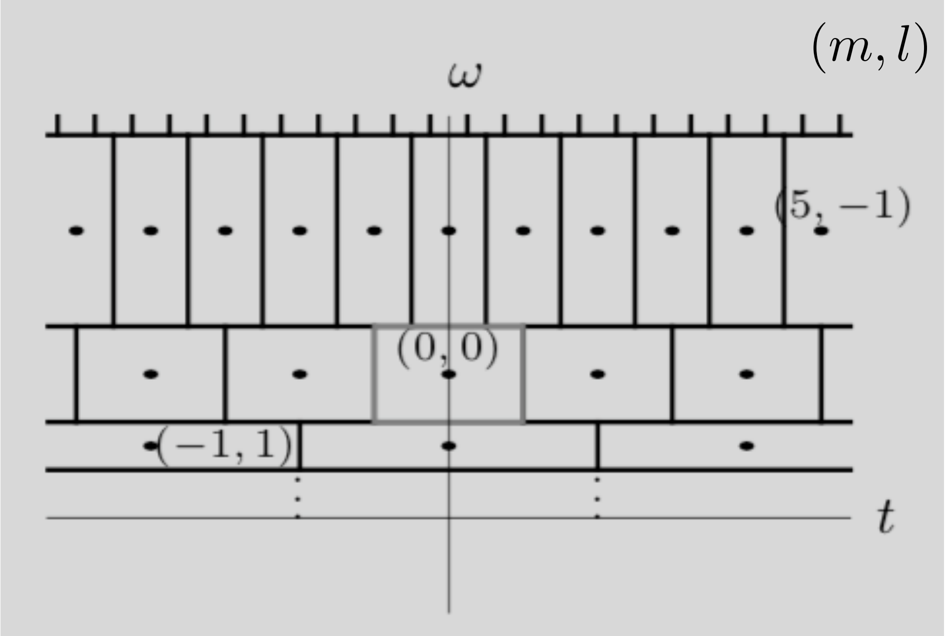
Time-frequency representation

time shift and modulation



$$\varphi_{k,m}(t) = e^{jk\omega_0 t}\varphi(t - mt_0) \quad k, m \in \mathbb{Z}$$

time shift and scaling

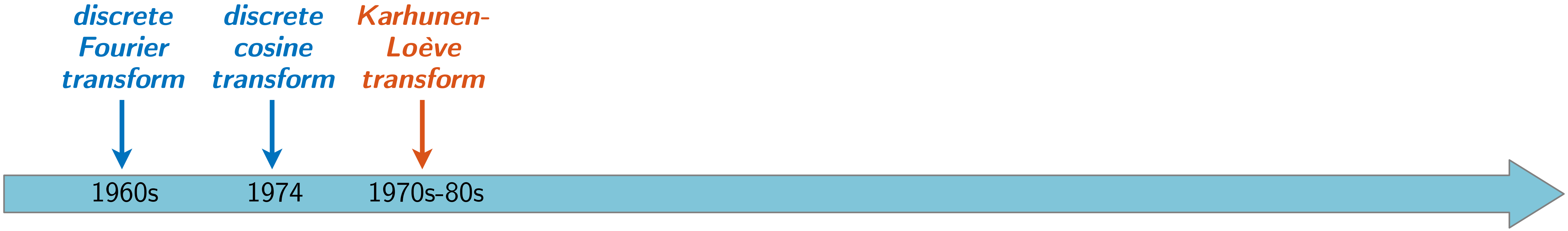


$$\varphi_{l,m}(t) = \boxed{\varphi(a^{-l}t - mt_0)} \quad l, m \in \mathbb{Z}$$

$$\varphi(a^{-l}(t - ma^l t_0))$$

Vetterli et al., "Foundations of signal processing," Cambridge University Press, 2014.

1970s-80s: STFT



discrete Fourier transform

1960s

discrete cosine transform

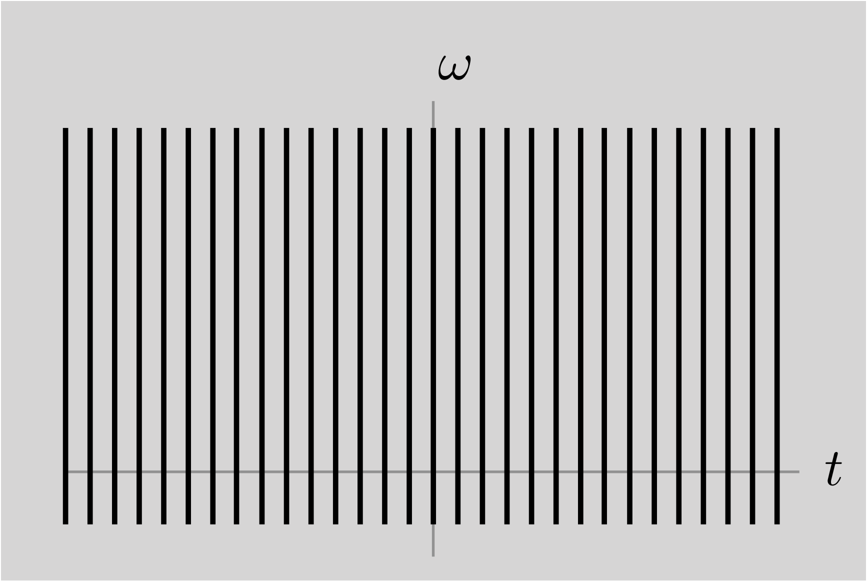
1974

Karhunen-Loève transform

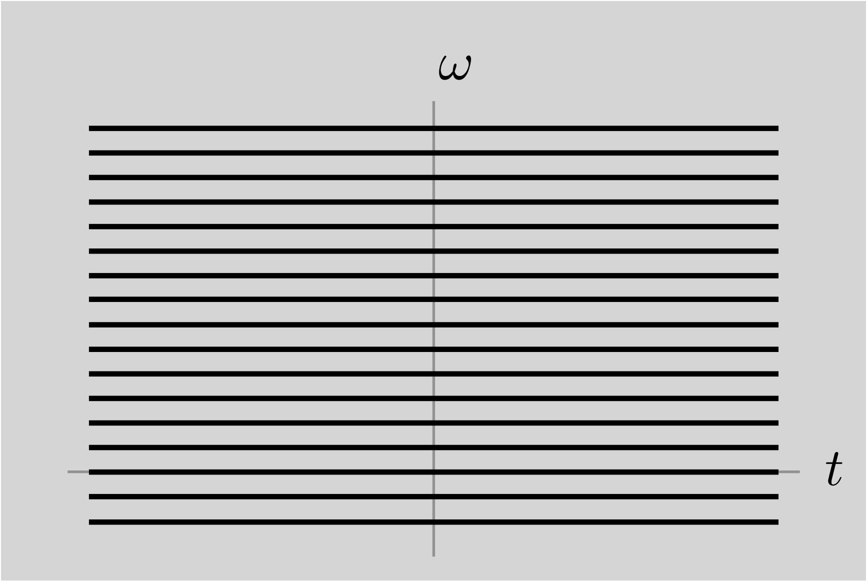
1970s-80s

- delta functions are not localised in frequency
- Fourier basis functions (complex exponentials) are not localised in time

time-domain



frequency-domain



1970s-80s: STFT

discrete Fourier transform 1960s

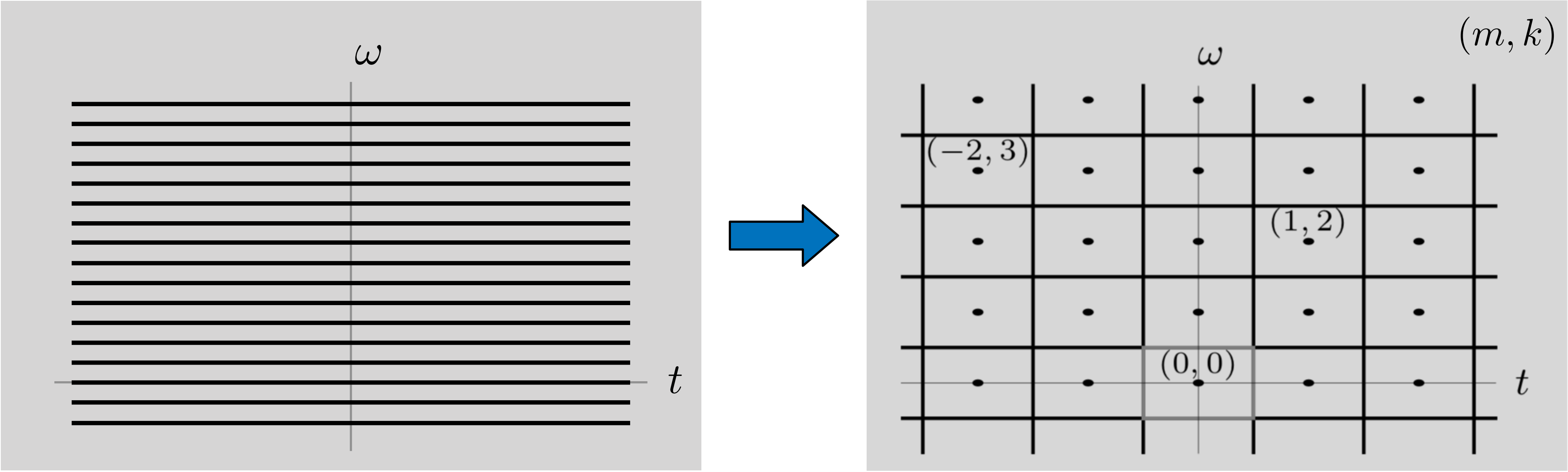
discrete cosine transform 1974

Karhunen-Loève transform 1970s-80s

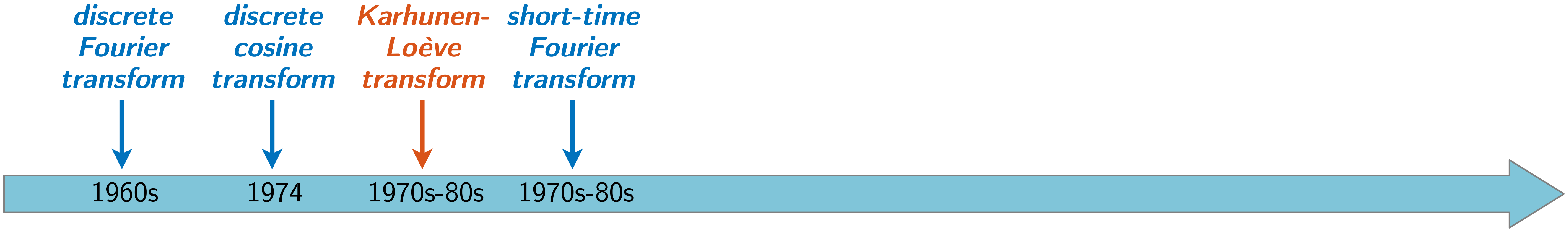
short-time Fourier transform 1970s-80s

- consider a set of shifted and modulated versions of a low-pass function

$$\varphi_{k,m}(t) = e^{jk\omega_0 t}\varphi(t - mt_0) \quad k, m \in \mathbb{Z}$$



1970s-80s: STFT



discrete Fourier transform

1960s

discrete cosine transform

1974

Karhunen-Loève transform

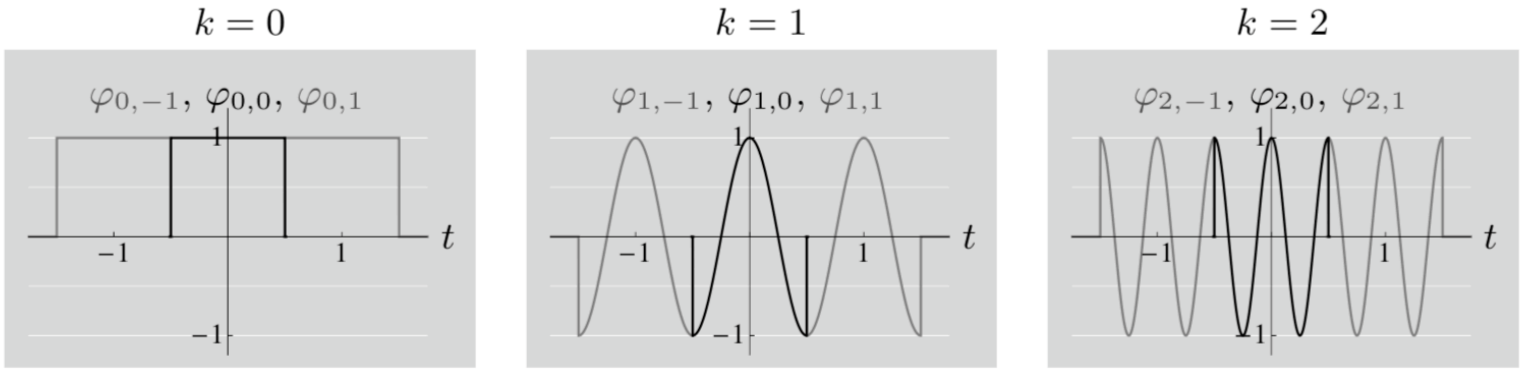
1970s-80s

short-time Fourier transform

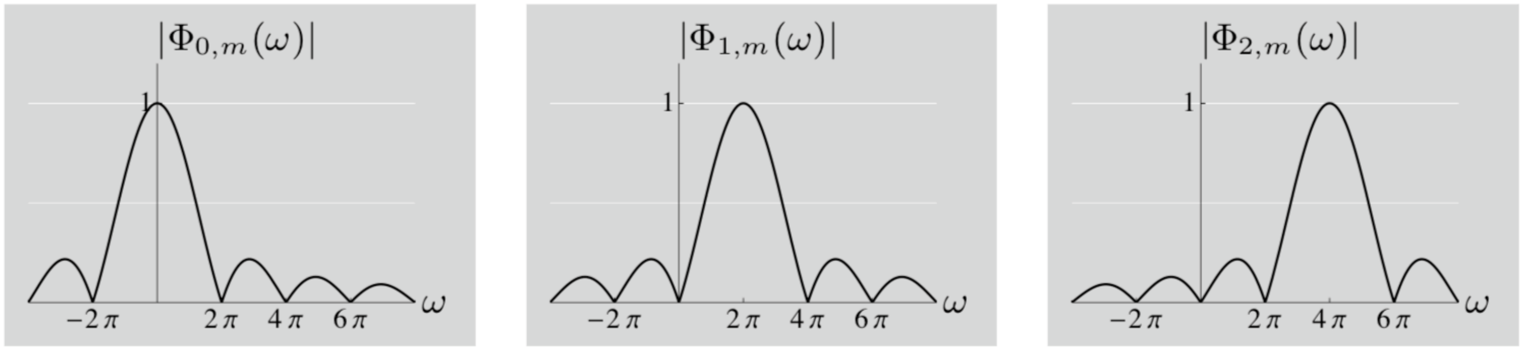
1970s-80s

- example: consider a box function and $t_0 = 1$, $\omega_0 = 2\pi$

$$\varphi_{k,m}(t) = e^{jk2\pi t}\varphi(t-m), \quad \varphi(t) = \begin{cases} 1, & \text{for } |t| \leq \frac{1}{2}; \\ 0, & \text{otherwise.} \end{cases}$$



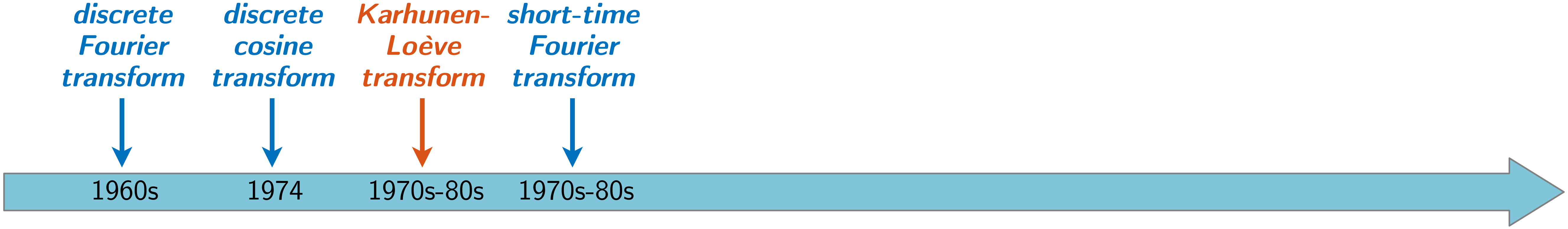
Basis functions (real parts only).



Magnitudes of the Fourier transform.

Vetterli et al., “Foundations of signal processing,” Cambridge University Press, 2014.

1970s-80s: STFT


discrete Fourier transform
1960s
discrete cosine transform
1974
Karhunen-Loève transform
1970s-80s
short-time Fourier transform
1970s-80s

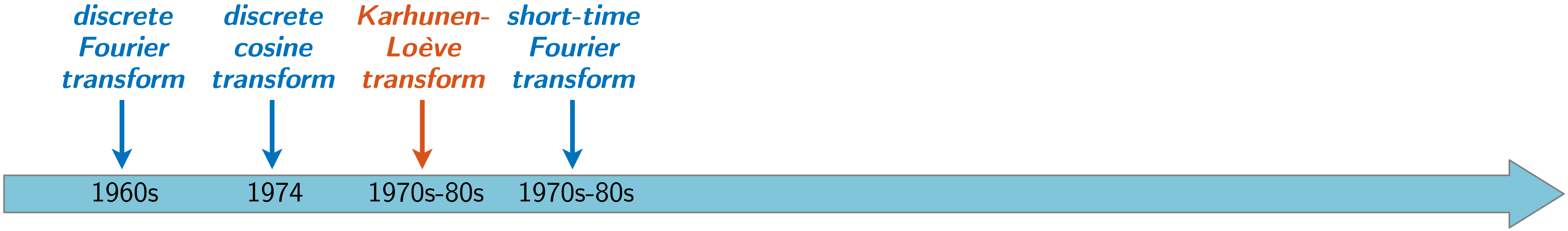
- we can define the following transform

$$X_{k,m} = \int_{-\infty}^{\infty} x(t)\varphi_{k,m}^{*}(t)dt = \int_{-\infty}^{\infty} x(t)\varphi(t-mt_0)e^{-jk\omega_0 t}dt$$

$$\Rightarrow \quad \boxed{X(\omega,\tau) = \int_{-\infty}^{\infty} x(t)\varphi_{\omega,\tau}^{*}(t)dt = \int_{-\infty}^{\infty} x(t)\varphi(t-\tau)e^{-j\omega t}dt}$$

- applying **time-localised** window to the signal before taking Fourier transform: windowed or short-time Fourier transform (STFT)
- Gaussian window achieves **localisation in frequency**: Gabor transform
- STFT maps a 1-D function into a 2-D function (highly **over-complete**)

DCT vs STFT



discrete Fourier transform 1960s

discrete cosine transform 1974

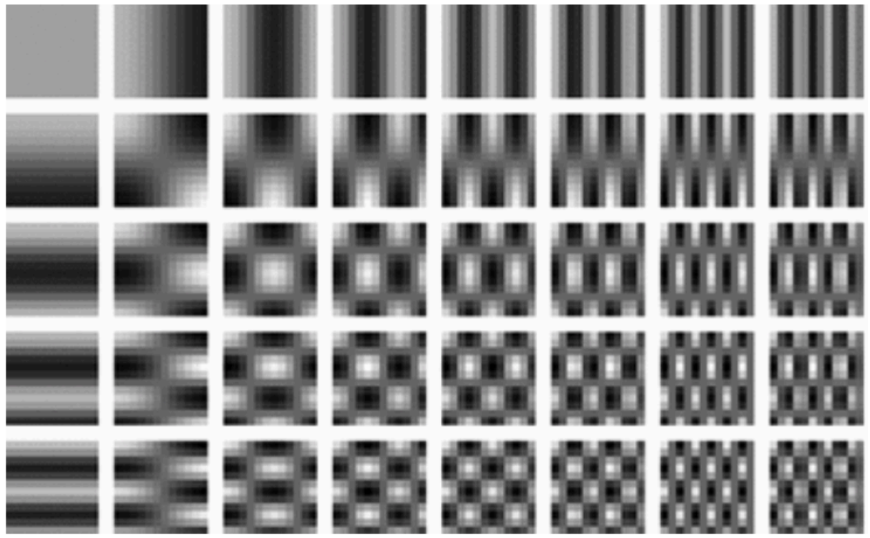
Karhunen-Loève transform 1970s-80s

short-time Fourier transform 1970s-80s

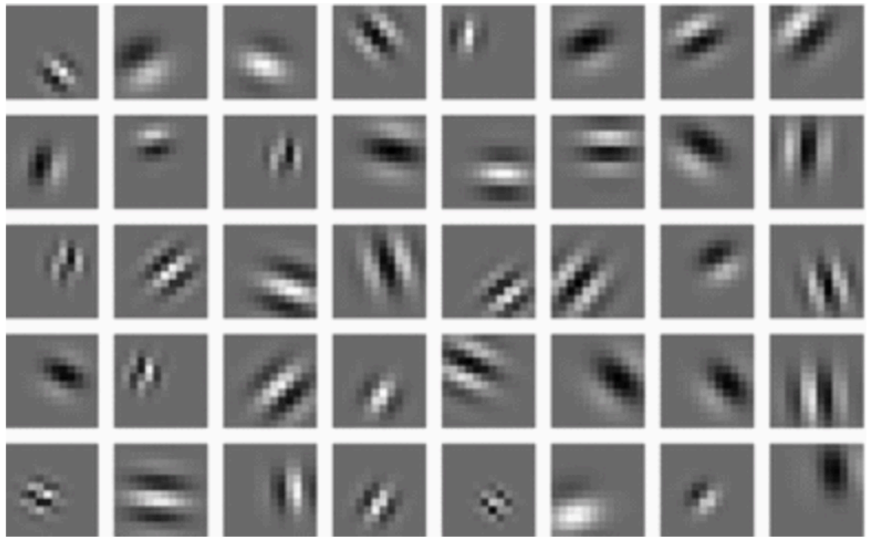
- discrete STFT provides a **non-orthogonal** and **over-complete** dictionary

$$\phi_{k,m}(n) = e^{j\frac{2\pi}{N}nk}\varphi(n-mN)$$

DCT

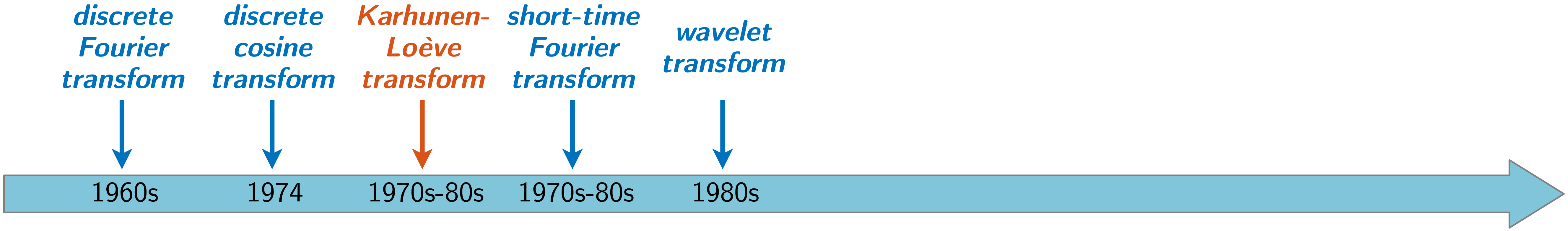


STFT



Rubinstein et al., "Dictionaries for sparse representation modeling," Proceedings of the IEEE, 2010.

1980s: wavelet transform



discrete Fourier transform

discrete cosine transform

Karhunen-Loève transform

short-time Fourier transform

wavelet transform

1960s

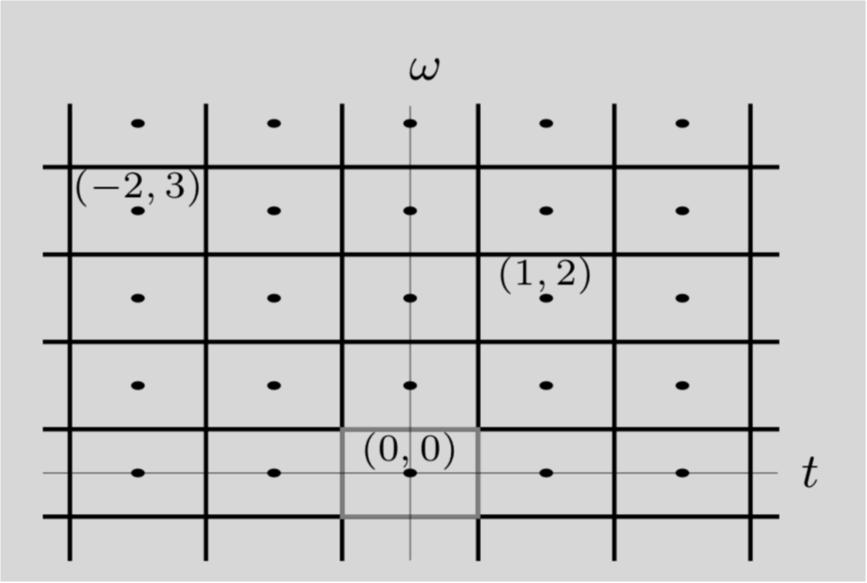
1974

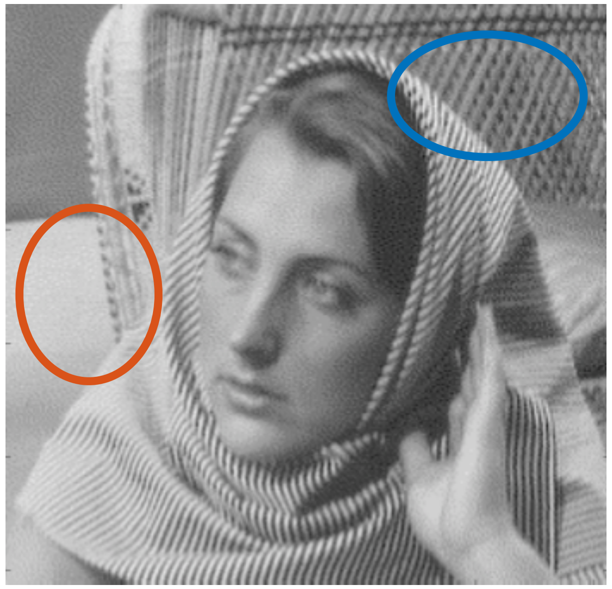
1970s-80s

1970s-80s

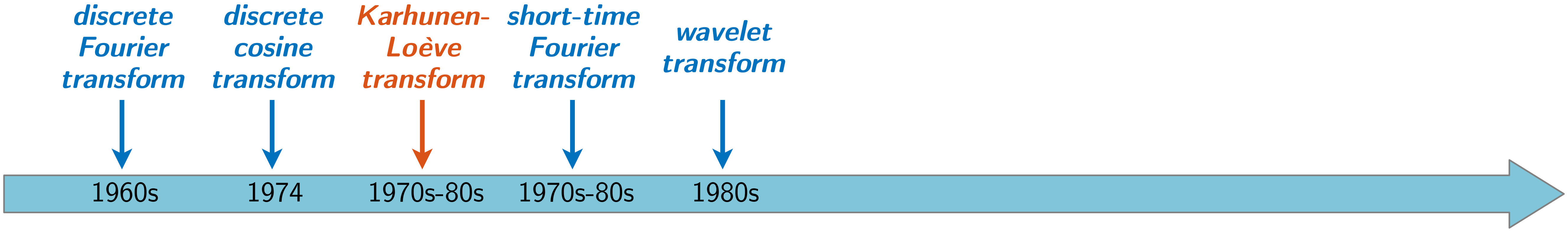
1980s

- STFT atoms have **fixed time-frequency resolution**
- often times a **multi-resolution** representation is needed to capture various scales in natural signals





1980s: wavelet transform



discrete Fourier transform — 1960s

discrete cosine transform — 1974

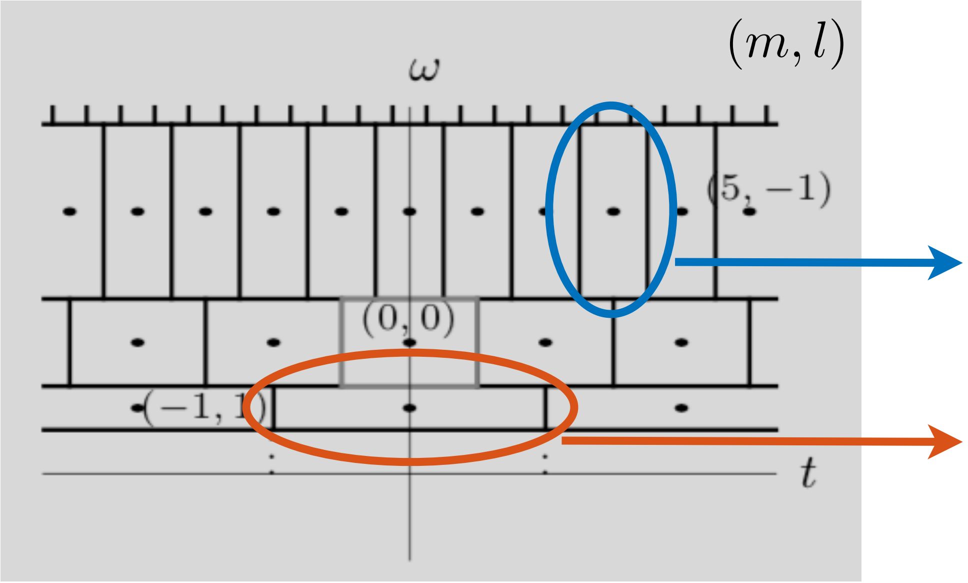
Karhunen-Loève transform — 1970s-80s

short-time Fourier transform — 1970s-80s

wavelet transform — 1980s

- consider a set of shifted and scaled versions of a band-pass function

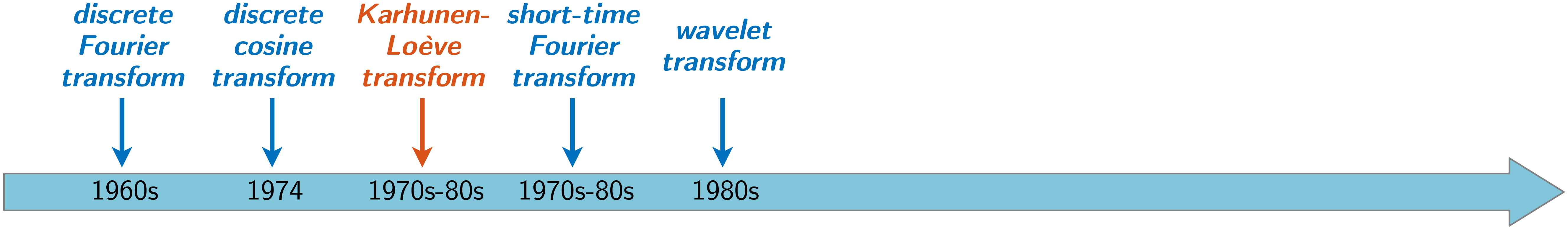
$$\varphi_{l,m}(t) = \varphi(a^{-l}t - mt_0) = \varphi(\frac{t - ma^l t_0}{a^l}) \quad l, m \in \mathbb{Z}$$



good time resolution (short-term)

good frequency resolution (long-term)

1980s: wavelet transform


discrete Fourier transform
1960s

discrete cosine transform
1974

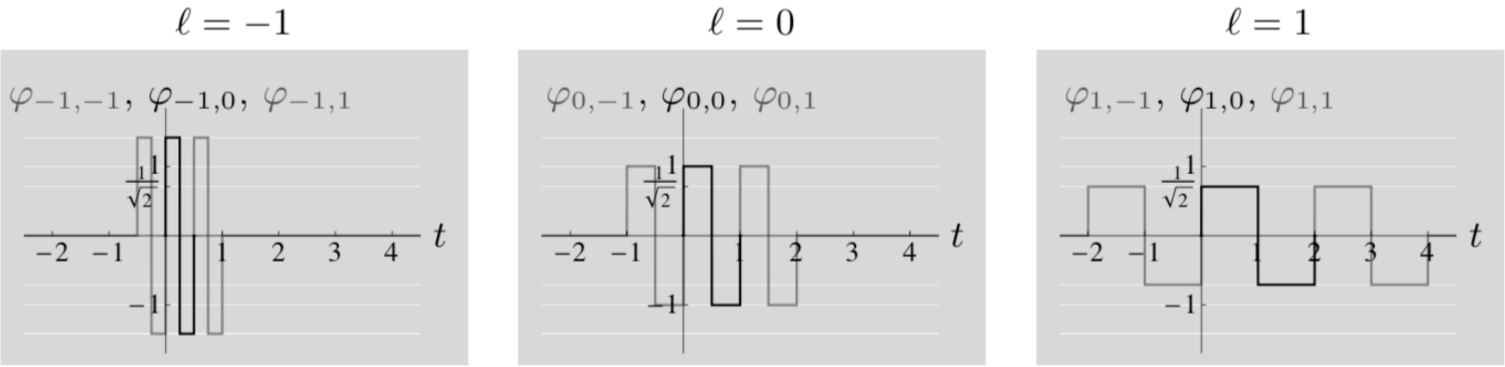
Karhunen-Loève transform
1970s-80s

short-time Fourier transform
1970s-80s

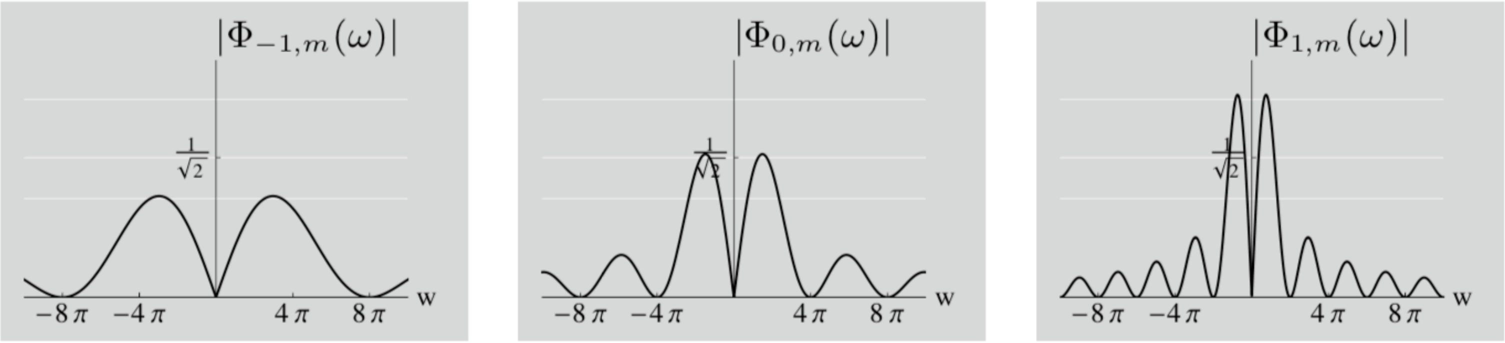
wavelet transform
1980s

- example: consider a square wave function and $t_0 = 1$, $a = 2$

$$\varphi_{l,m}(t) = \varphi(\frac{t - 2^l m}{2^l}), \quad \varphi(t) = \begin{cases} 1, & \text{for } 0 \le t < \frac{1}{2}; \\ -1, & \text{for } \frac{1}{2} \le t < 1; \\ 0, & \text{otherwise.} \end{cases}$$



Basis functions.



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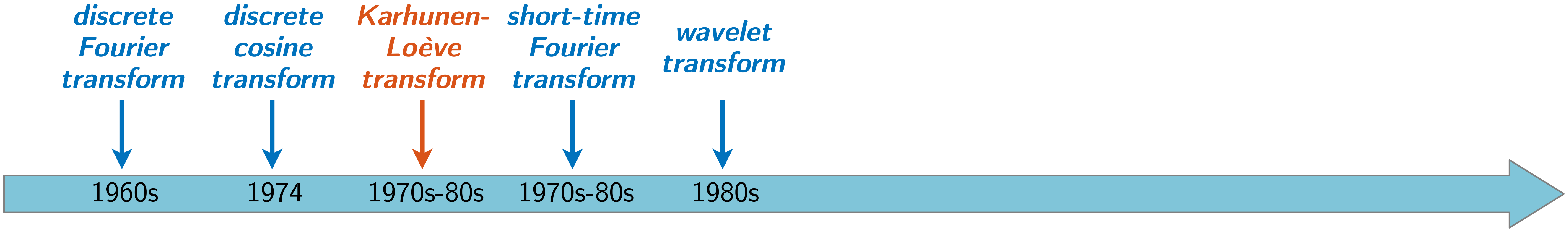
Vetterli et al., "Foundations of signal processing," Cambridge University Press, 2014.

1980s: wavelet transform



discrete Fourier transform — 1960s

discrete cosine transform — 1974

Karhunen-Loève transform — 1970s-80s

short-time Fourier transform — 1970s-80s

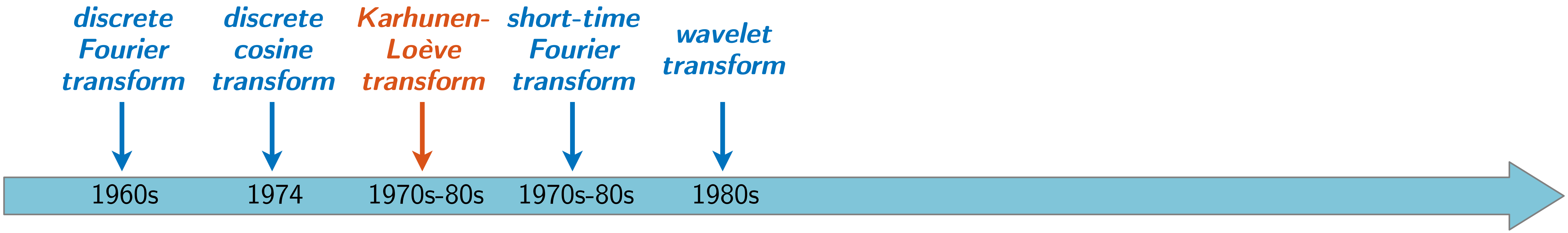
wavelet transform — 1980s

- consider a more general function and define the following transform

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\boxed{\psi(\frac{t-\tau}{s})} \Rightarrow \boxed{X(s,\tau) = \int_{-\infty}^{\infty} x(t)\psi^*_{s,\tau}(t)dt = \int_{-\infty}^{\infty} x(t)\frac{1}{\sqrt{s}}\psi^*(\frac{t-\tau}{s})dt}$$

- the prototype function $\psi(t)$
  - has a compact support (small or “-let”)
  - is band-pass with zero mean (“wave”): $\int_{-\infty}^{\infty} \psi(t)dt = 0$
- this is called the **continuous wavelet transform (CWT)**
- CWT maps a 1-D function into a 2-D function (highly **over-complete**)

1980s: wavelet transform



discrete Fourier transform

1960s

discrete cosine transform

1974

Karhunen-Loève transform

1970s-80s

short-time Fourier transform

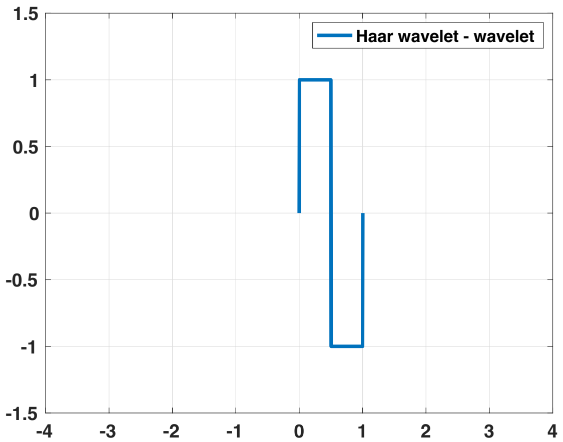
1970s-80s

wavelet transform

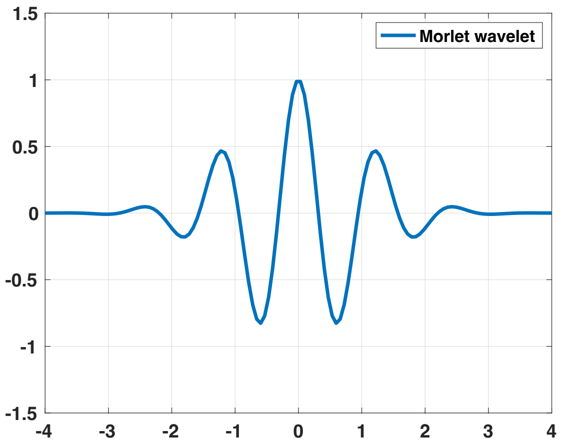
1980s

- examples of prototype function (mother wavelet)

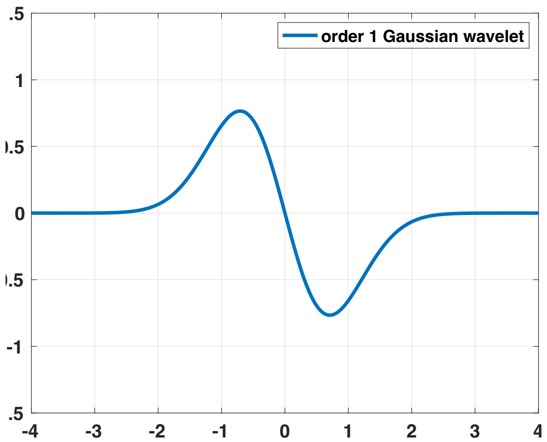
Haar



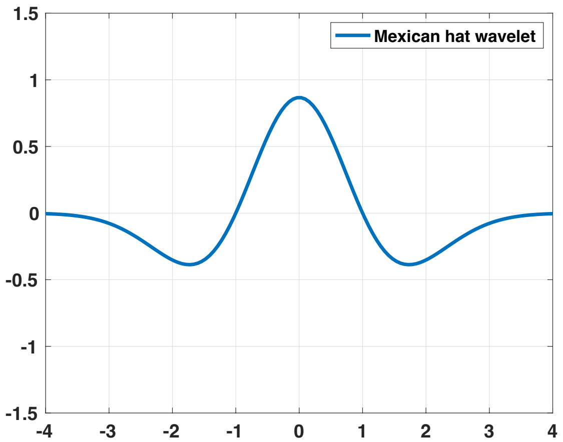
Morlet



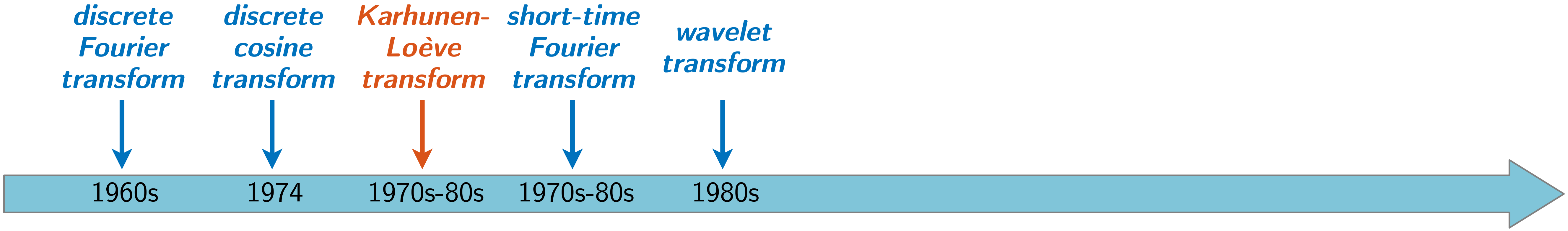
Derivative of Gaussian



Marr (Mexican hat)



1980s: wavelet transform


discrete Fourier transform
1960s
discrete cosine transform
1974
Karhunen-Loève transform
1970s-80s
short-time Fourier transform
1970s-80s
wavelet transform
1980s

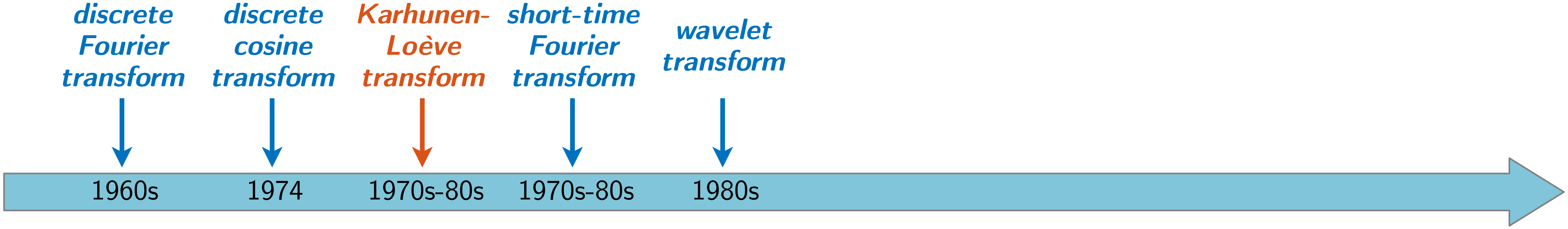
- CWT can be discretised to define a discrete-time wavelet transform (DTWT), but both CWT and DTWT are **non-orthogonal** and **over-complete**
- however, unlike STFT, we can design an **orthogonal** wavelet transform through a **multi-resolution analysis**

$$\psi_{l,m}(t) = \frac{1}{\sqrt{2^l}} \boxed{\psi} \left(\frac{t - 2^l m}{2^l} \right)$$

design principle

- functions at given scale l form an orthogonal basis of a space at scale l
- all functions across different scales are also orthogonal

1980s: wavelet transform



discrete Fourier transform

1960s

discrete cosine transform

1974

Karhunen-Loève transform

1970s-80s

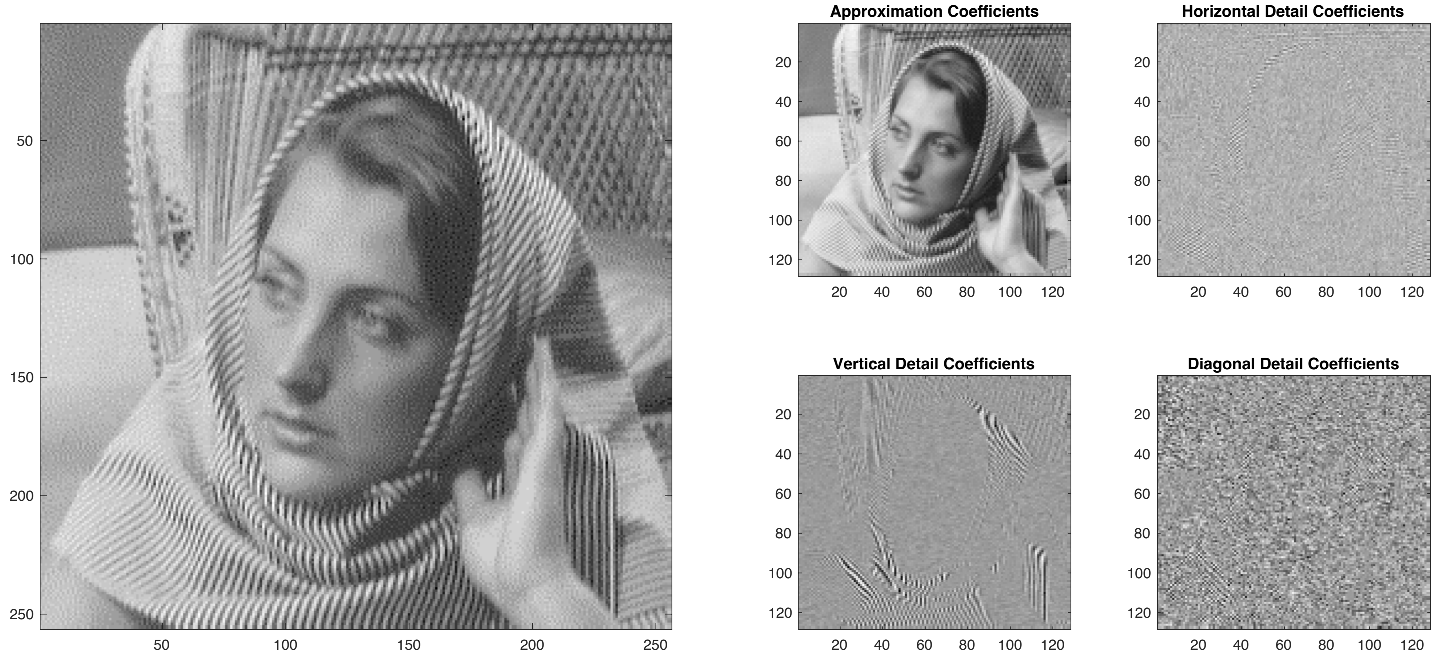
short-time Fourier transform

1970s-80s

wavelet transform

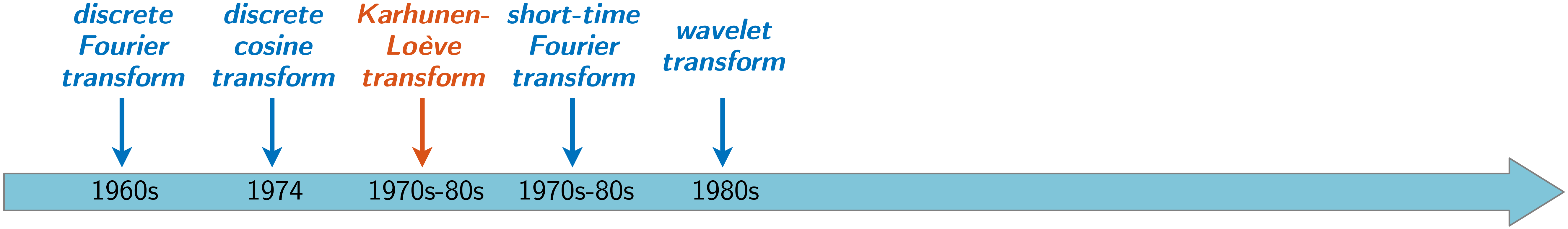
1980s

- the discrete version is the discrete wavelet transform (DWT) which provides an **orthogonal** dictionary



- DWT is behind the JEPG 2000 image compression standard

DCT vs KLT vs STFT vs DWT



discrete Fourier transform

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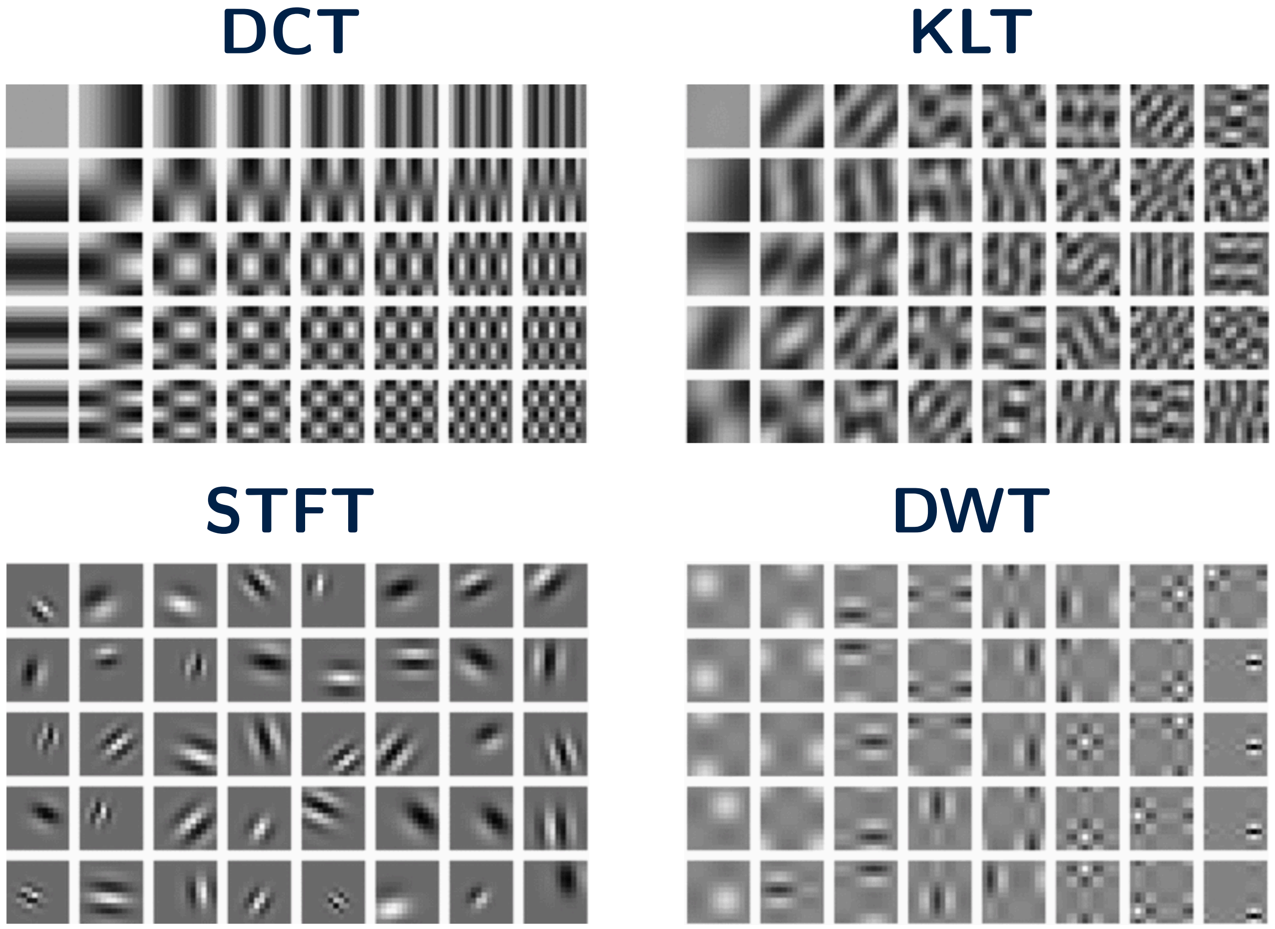
1970s-80s

wavelet transform

1980s

- comparison of the dictionaries we looked at so far



DCT

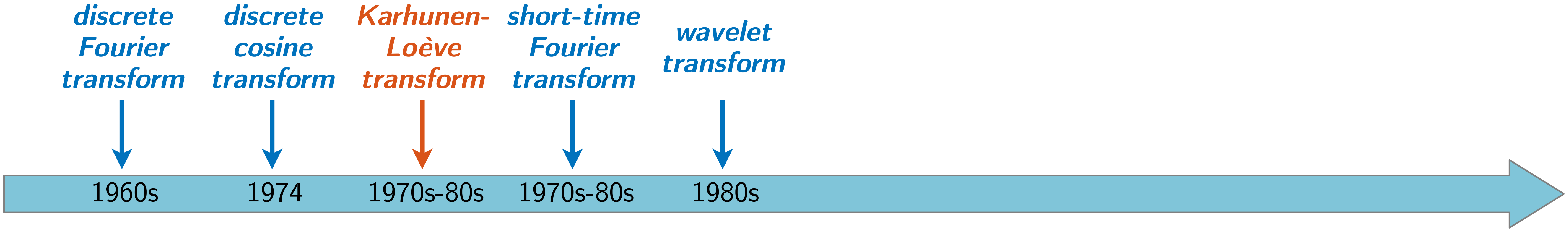
KLT

STFT

DWT

Rubinstein et al., "Dictionaries for sparse representation modeling," Proceedings of the IEEE, 2010.

Transform/analytic dictionary design


discrete Fourier transform
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discrete cosine transform
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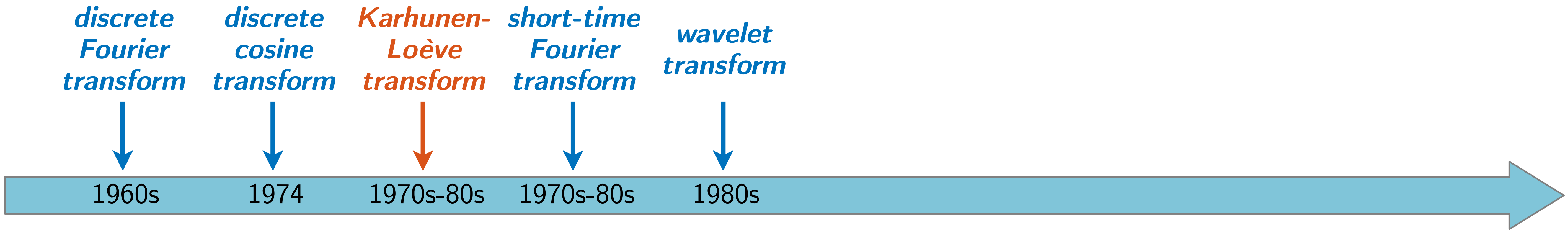
short-time Fourier transform
1970s-80s
wavelet transform
1980s

- summary
  - modelling data by a simpler class of mathematical functions
    - smooth functions (DFT, DCT)
    - piecewise-smooth functions (wavelets)
  - desired properties
    - localisation (STFT, wavelets)
    - multi-resolution (wavelets)
    - adaptivity (KLT, wavelet packets)
  - fast implementation is usually available
  - limited expressiveness

Outline

- A historical overview of dictionary design techniques
  - signal representation via stochastic models
  - transforms & analytic dictionaries
  - trained dictionaries (dictionary learning)
- Discussion
  - applications
  - connection with deep learning

A paradigm shift in dictionary design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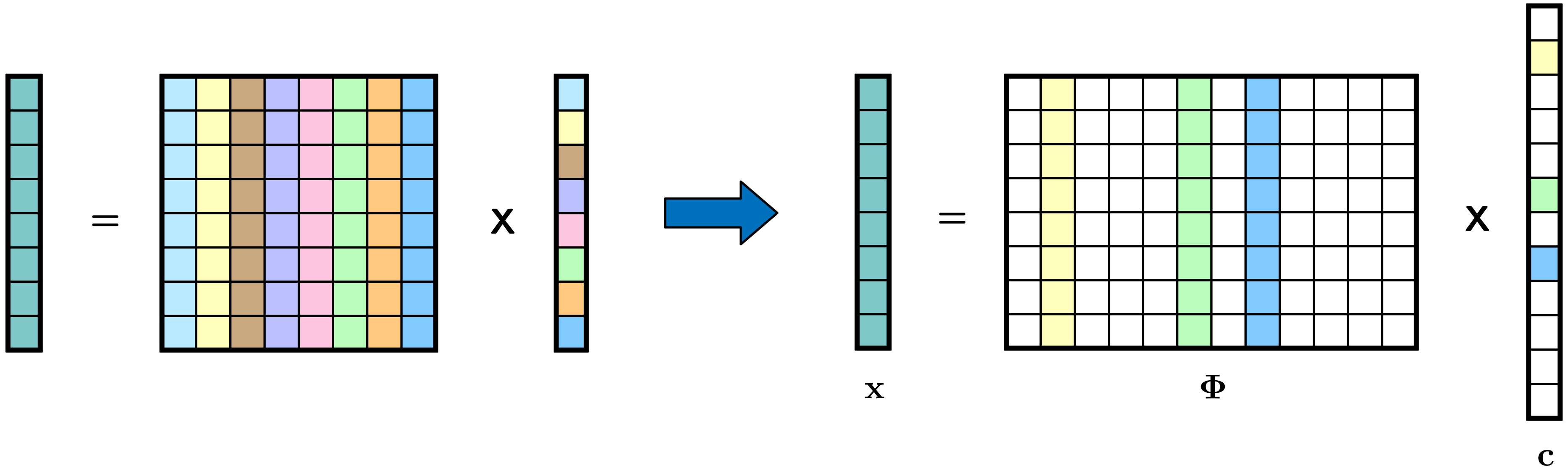
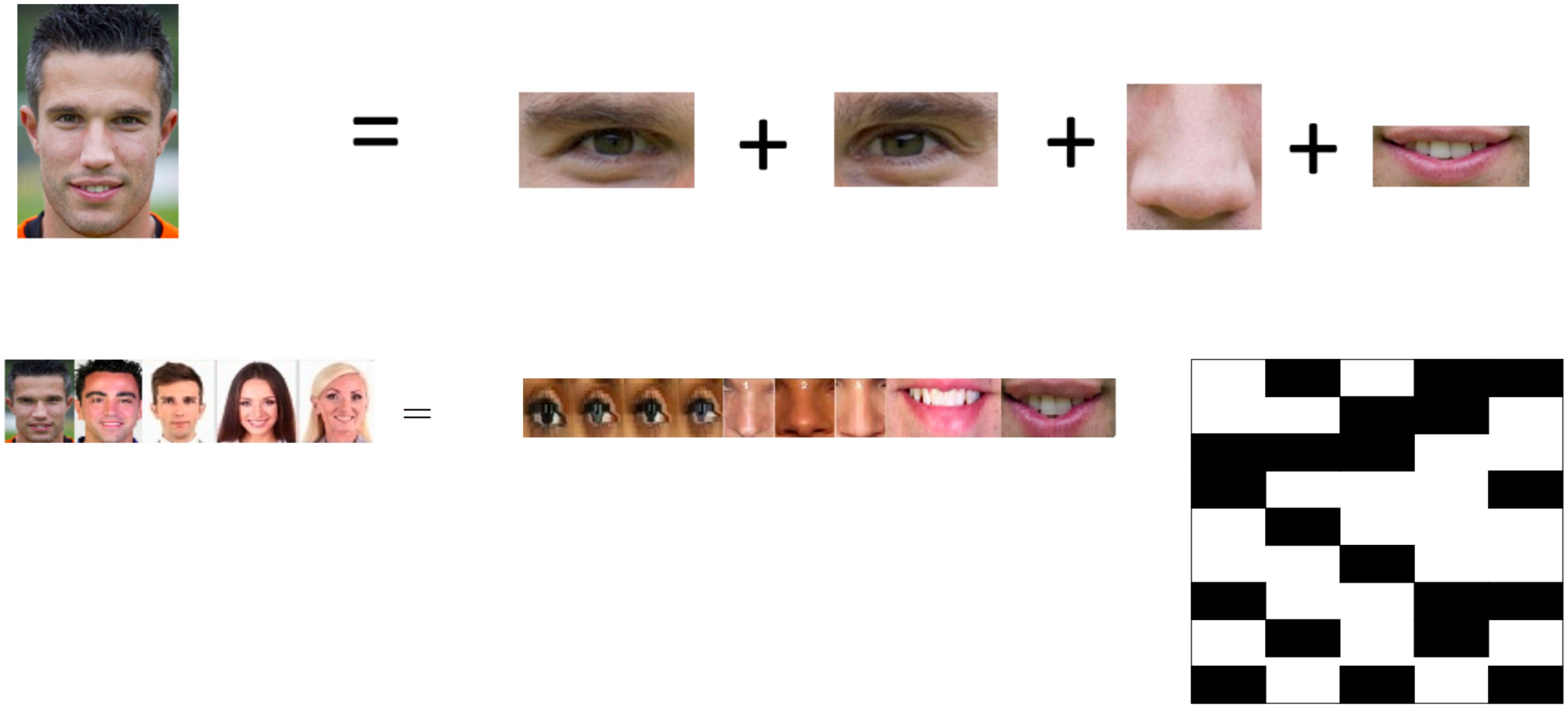


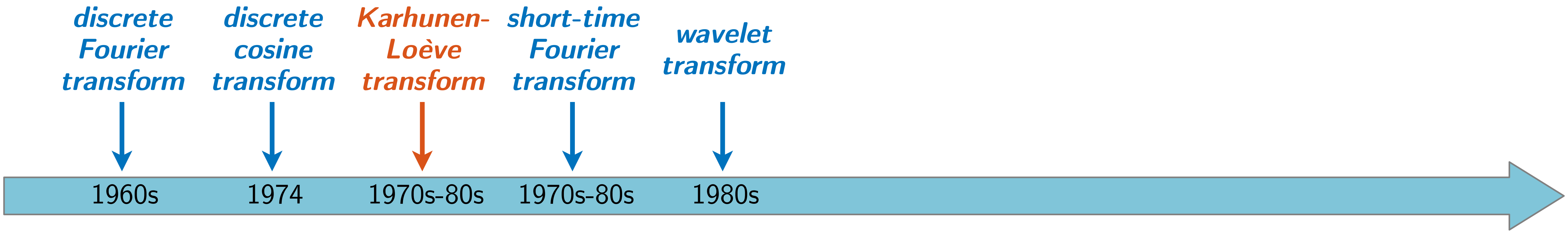
orthogonal atoms	non-orthogonal atoms
complete dictionary	over-complete dictionary
all signals use all atoms	different signals use different atoms
dense coefficients	sparse coefficients
mathematical modelling	adaptation to data realisations

Illustrative example

- **Modelling assumption:** Each data point is a combination of only a few (sparse) fundamental elements, i.e., dictionary atoms



Sparse representations



discrete Fourier transform — 1960s

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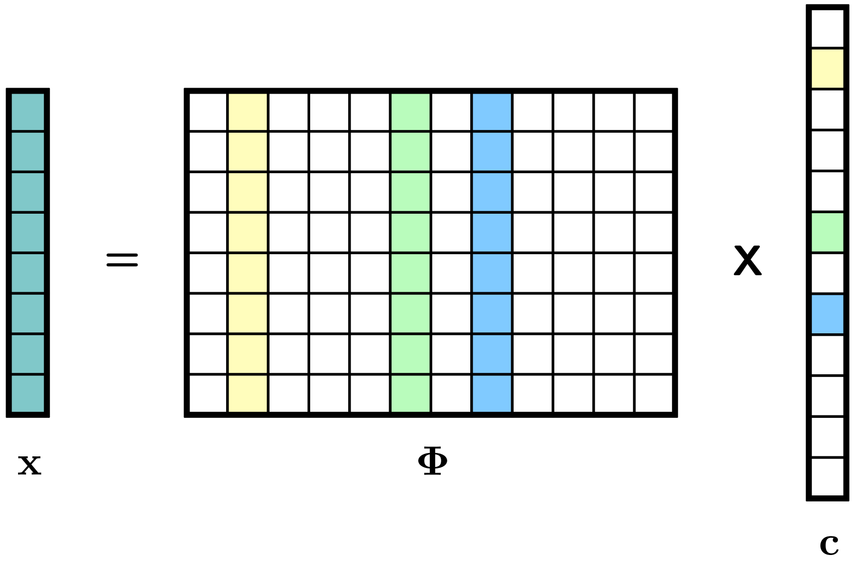


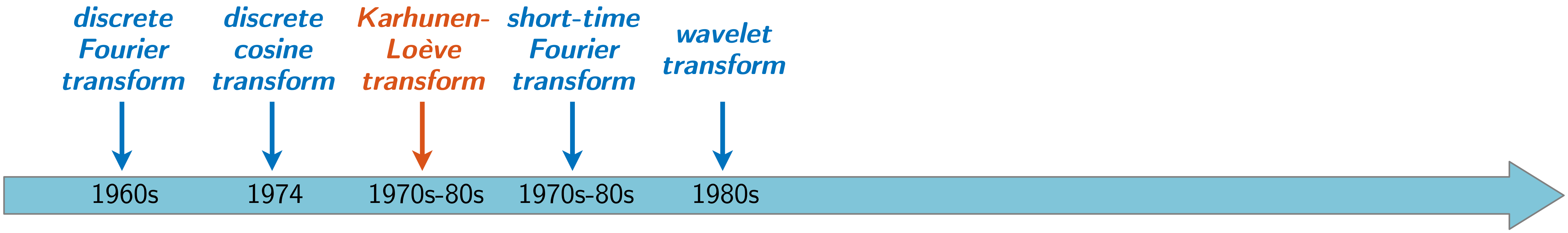
- given dictionary, express signal as linear combination of a small number of atoms

$$\min_{\mathbf{c}} ||\mathbf{c}||_0 \ \text{ subject to } \ \mathbf{x} = \boldsymbol{\Phi}\mathbf{c} + \boldsymbol{\eta} \ \text{ and } \ ||\boldsymbol{\eta}||_2^2 \leq \epsilon$$

- the problem is NP-hard
- two approximation algorithms
  - matching pursuit (MP)
  - least absolute shrinkage and selection operator (Lasso)

Mallat and Zhang, "Matching pursuits with time-frequency dictionaries," IEEE TSP, 1993.
Tibshirani, "Regression Shrinkage and Selection via the Lasso," Journal of the Royal Statistical Society: Series B, 1996.

Sparse representations



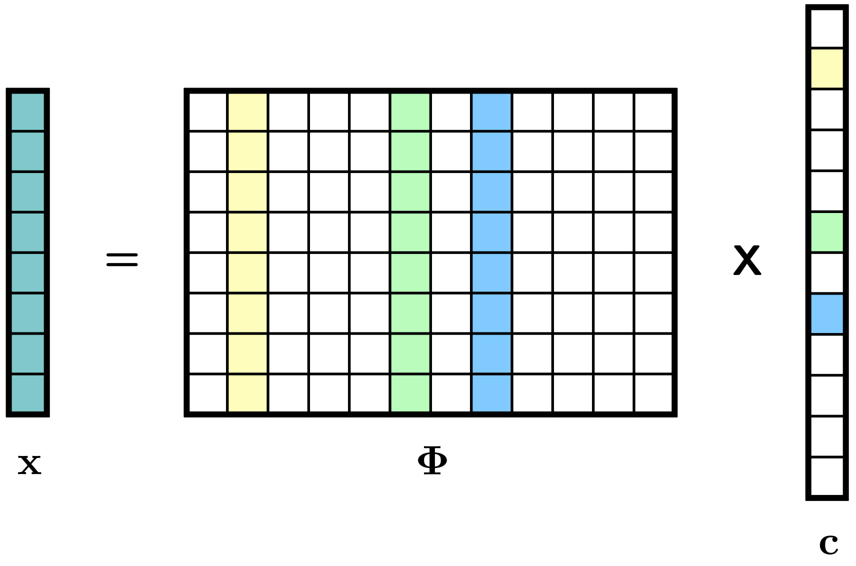
discrete Fourier transform 1960s

discrete cosine transform 1974

Karhunen-Loève transform 1970s-80s

short-time Fourier transform 1970s-80s

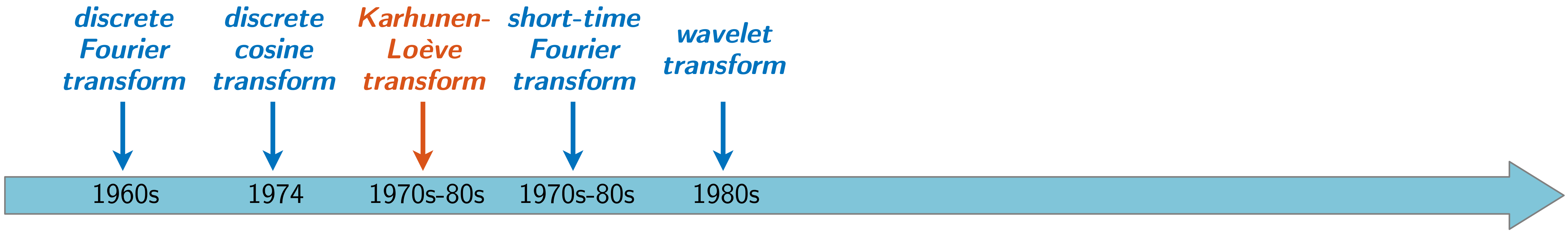
wavelet transform 1980s



- MP
  - choose a subset of atoms from $\mathbf{\Phi}$
  - one atom at a time to maximally (greedily) reduce approximation error
- Lasso
  - solve a convex relaxation by replacing the 0-norm with 1-norm on **c**

$$\min_{\mathbf{c}} ||\mathbf{x} - \mathbf{\Phi}\mathbf{c}||_2^2 + \lambda||\mathbf{c}||_1$$

Sparse representations



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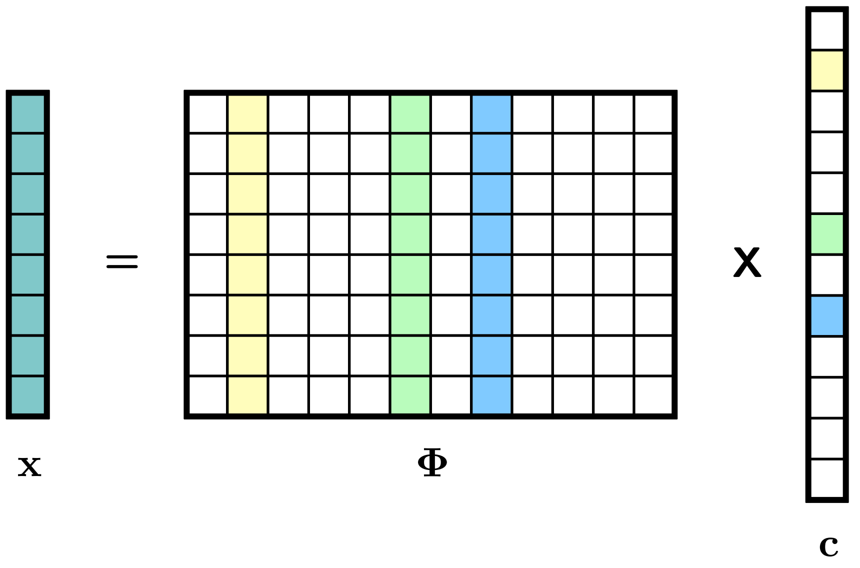
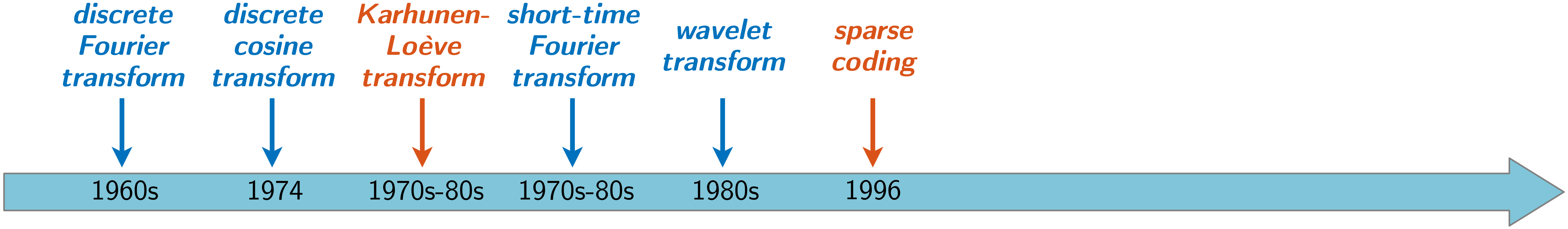
1980s



- given dictionary, MP and Lasso can find a sparse approximation of the data
- the sparsity depends on not only data but also the **dictionary**
- finding optimised dictionaries is the goal of **dictionary learning**

Dictionary learning: Probabilistic approach



discrete Fourier transform
1960s

discrete cosine transform
1974

Karhunen-Loève transform
1970s-80s

short-time Fourier transform
1970s-80s

wavelet transform
1980s

sparse coding
1996

- probabilistic approach: maximum likelihood

$$\mathbf{\Phi}^* = \arg\max_{\mathbf{\Phi}}[\log\ P(\mathbf{x}|\mathbf{\Phi})]$$

$$= \arg\max_{\mathbf{\Phi}}[\log \int_{\mathbf{c}} P(\mathbf{x}|\mathbf{c},\mathbf{\Phi})P(\mathbf{c})d\mathbf{c}]$$

⬇

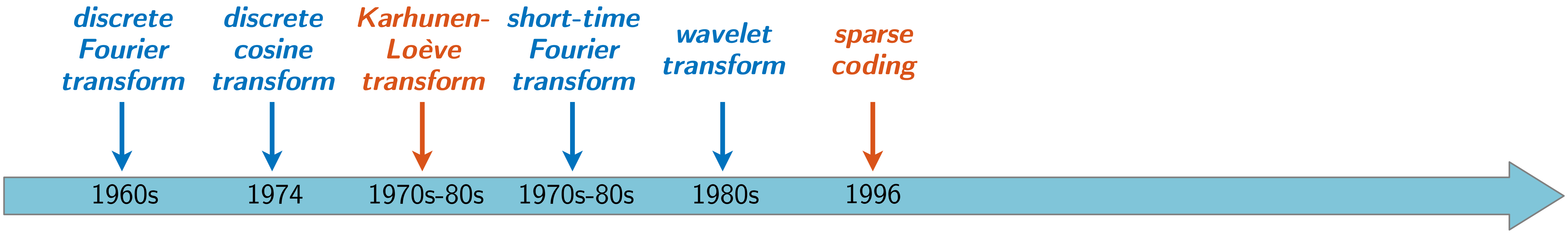
- assumption 1: Laplace distribution of coefficients $\mathbf{c}_i$
- assumption 2: Gaussian distribution of error $\boldsymbol{\eta}$

$$\mathbf{\Phi}^* = \arg\min_{\mathbf{\Phi},\mathbf{c}} -\log[P(\mathbf{x}|\mathbf{c},\mathbf{\Phi})P(\mathbf{c})]$$

$$= \boxed{\arg\min_{\mathbf{\Phi},\mathbf{c}} ||\mathbf{x}-\mathbf{\Phi}\mathbf{c}||_2^2 + \lambda||\mathbf{c}||_1}$$

Olshausen and Field, "Emergence of simple-cell receptive field properties by learning a sparse code for natural images," Nature, 1996.

Dictionary learning: Probabilistic approach



- the problem is solved by iterating between two steps

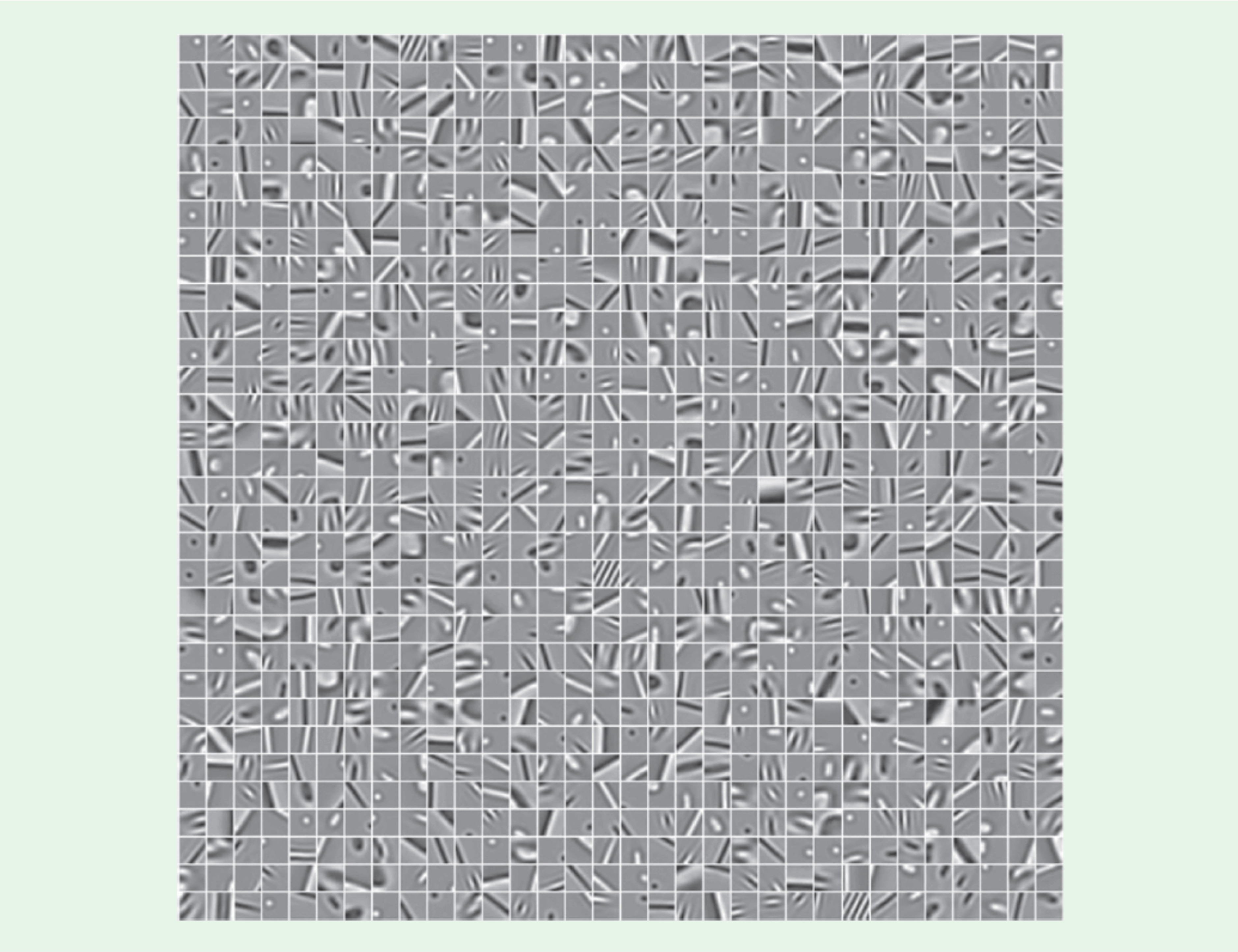
$$\min_{\mathbf{\Phi},\mathbf{c}} ||\mathbf{x} - \mathbf{\Phi}\mathbf{c}||_2^2 + \lambda||\mathbf{c}||_1$$

  - **sparse approximation:** given $\mathbf{\Phi}$, solve for **c** via Lasso
  - **dictionary update:** given **c**, update $\mathbf{\Phi}$ via gradient descent

- works at patch level for efficiency
- does not necessarily find global optimum
- trained atoms are remarkably similar to mammalian simple-cell receptive fields

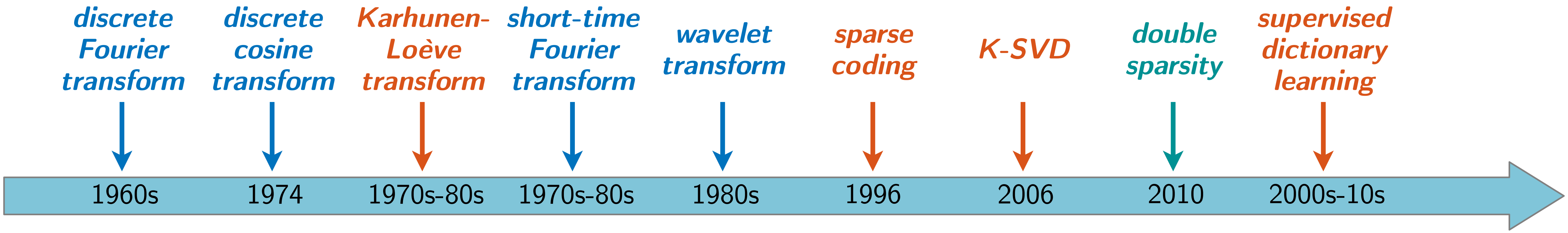
Olshausen and Field, "Emergence of simple-cell receptive field properties by learning a sparse code for natural images," Nature, 1996.

Dictionary learned with sparse coding



Olshausen et al., " "Learning real and complex overcomplete representations from the statistics of natural images," SPIE, 2009.

Dictionary learning



discrete Fourier transform 1960s

discrete cosine transform 1974

Karhunen-Loève transform 1970s-80s

short-time Fourier transform 1970s-80s

wavelet transform 1980s

sparse coding 1996

K-SVD 2006

double sparsity 2010

supervised dictionary learning 2000s-10s

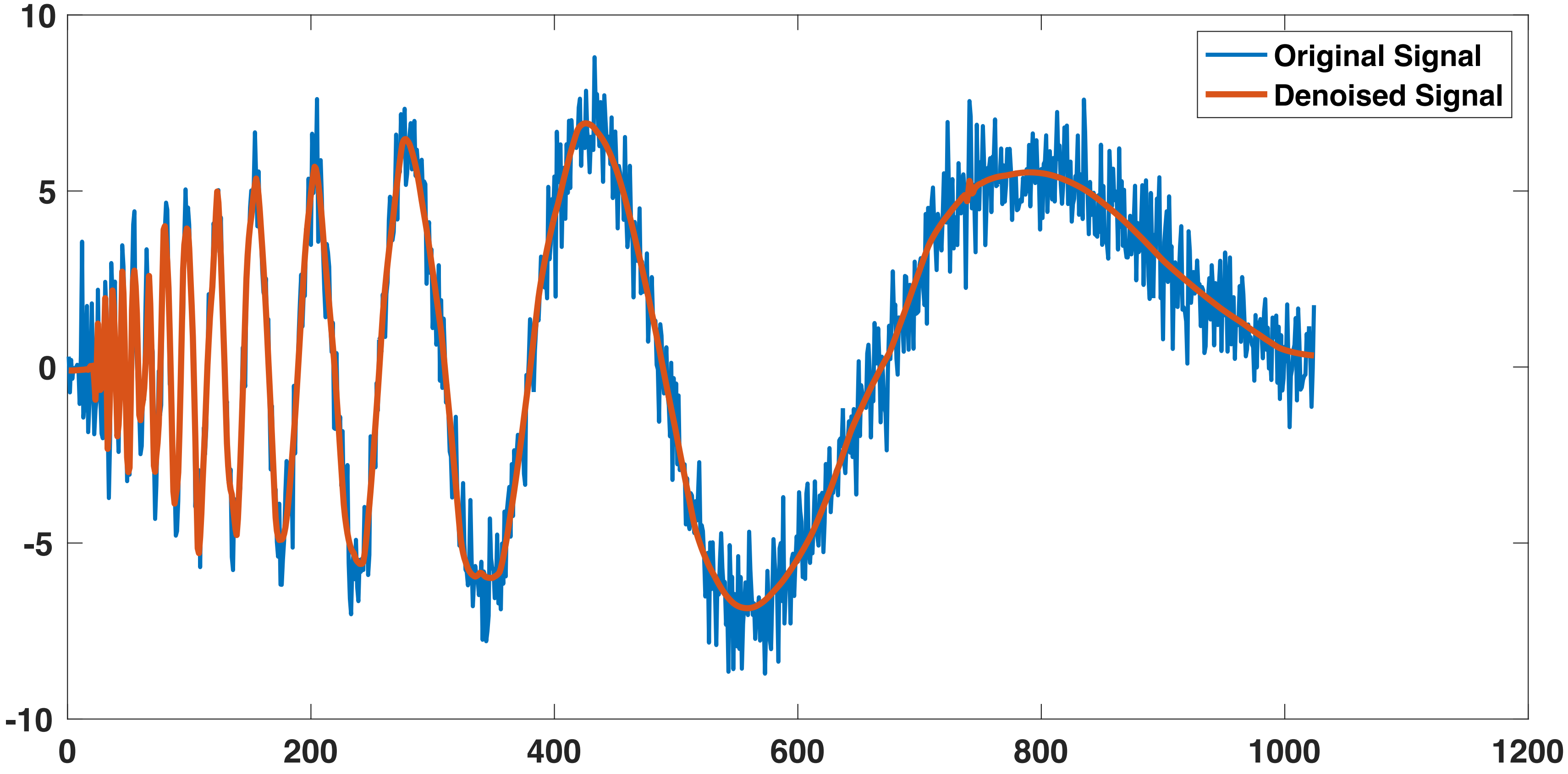
- summary
  - learning representations directly from data realisations
  - desired properties
    - over-completeness
    - sparse representations
    - efficiency in training
  - may be combined with analytical dictionary design
    - trained dictionary with structures (e.g., parametric dictionary learning)

Outline

- A historical overview of dictionary design techniques
  - signal representation via stochastic models
  - transforms & analytic dictionaries
  - trained dictionaries (dictionary learning)

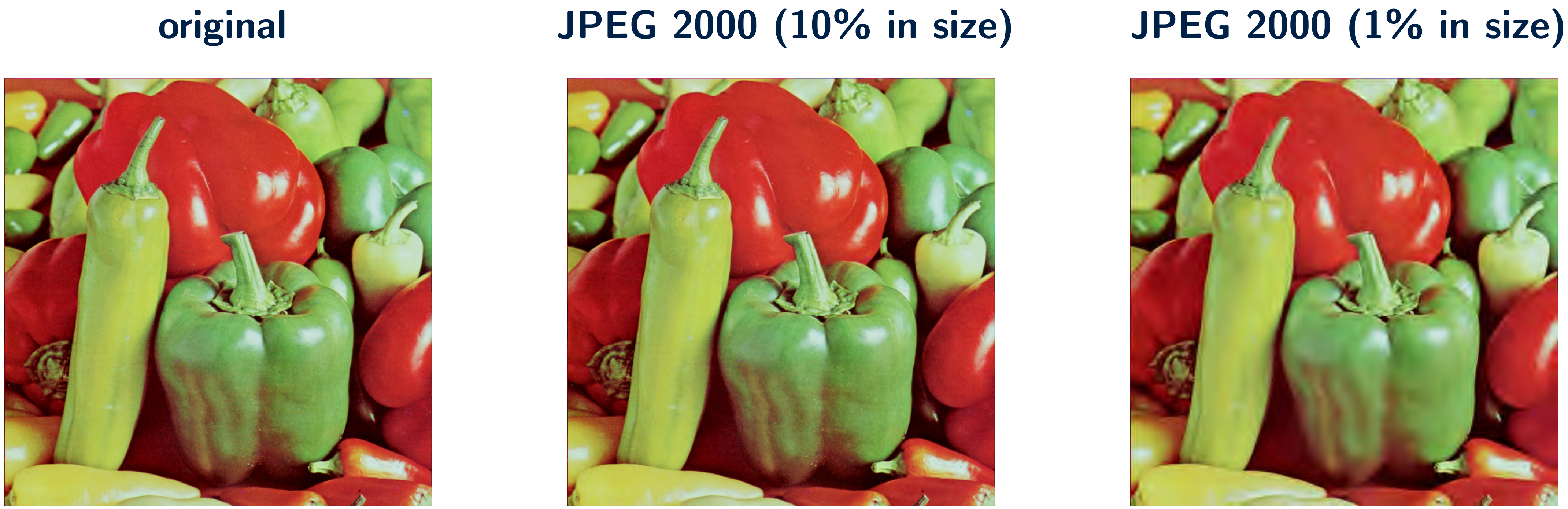
- Discussion
  - applications
  - connection with deep learning

Signal denoising



denoising using the order 4 symlets wavelets

Image compression



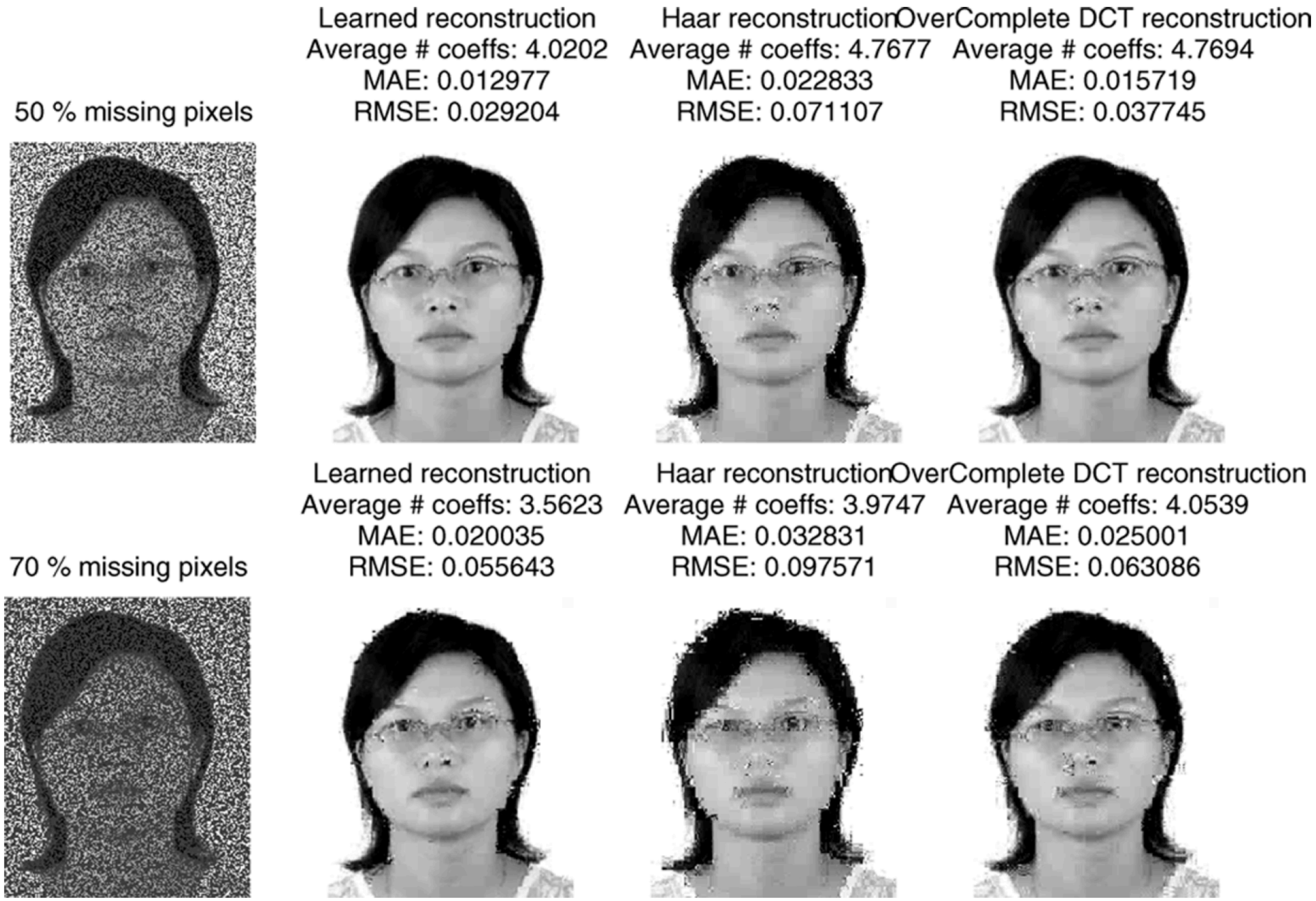
original

JPEG 2000 (10% in size)

JPEG 2000 (1% in size)

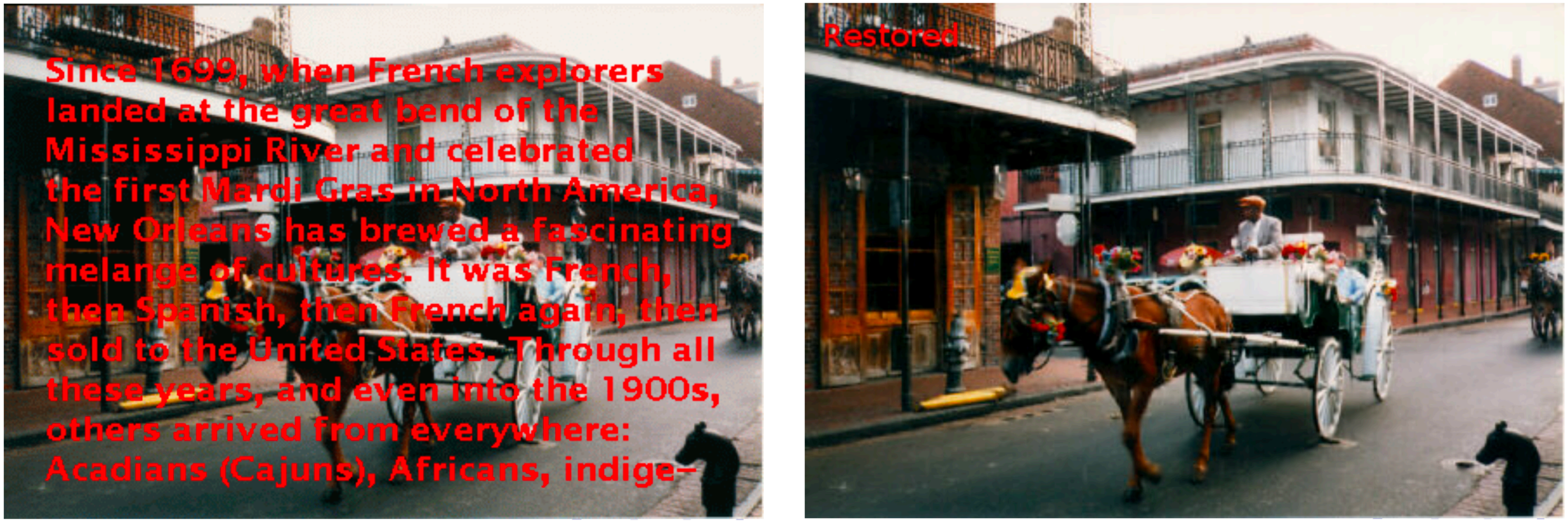
compression using the Cohen-Daubechies-Feauveau wavelets

Image reconstruction



Aharon et al., "K-SVD: An algorithm for designing overcomplete dictionaries for sparse representation," IEEE TSP, 2006.

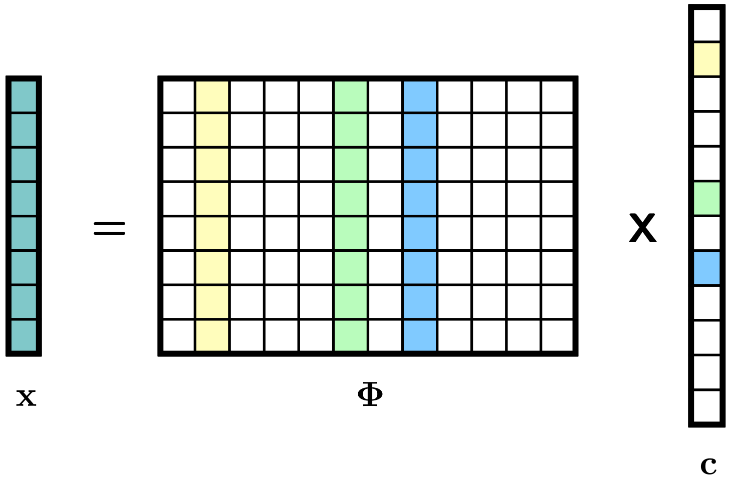
Image restoration

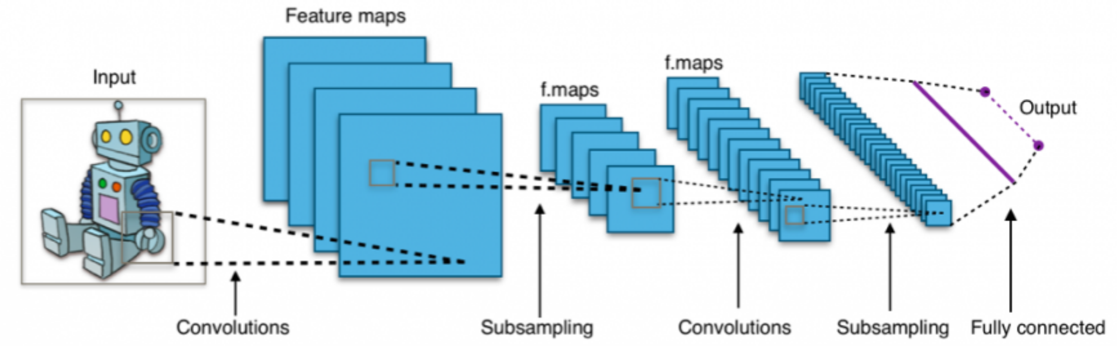

Since 1699, when French explorers
landed at the great bend of the
Mississippi River and celebrated
the first Mardi Gras in North America,
New Orleans has brewed a fascinating
melange of cultures. It was French,
then Spanish, then French again, then
sold to the United States. Through all
these years, and even into the 1900s,
others arrived from everywhere:
Acadians (Cajuns), Africans, indige-
Restored

Mairal et al., "Sparse representation for color image restoration," IEEE TIP, 2008.

Connection with deep learning

- Dictionary learning vs. Deep learning
  - both extract **feature representations** from data realisations
  - both apply **sparsifying operations** such as shrinkage or rectified linear units
  - the former leads to representations that are not necessarily **hierarchical** (shallow model, no convolution operator, no pooling)
  - the former is normally for **reconstruction/approximation** (similar to autoencoders) while the latter is mainly for **classification**

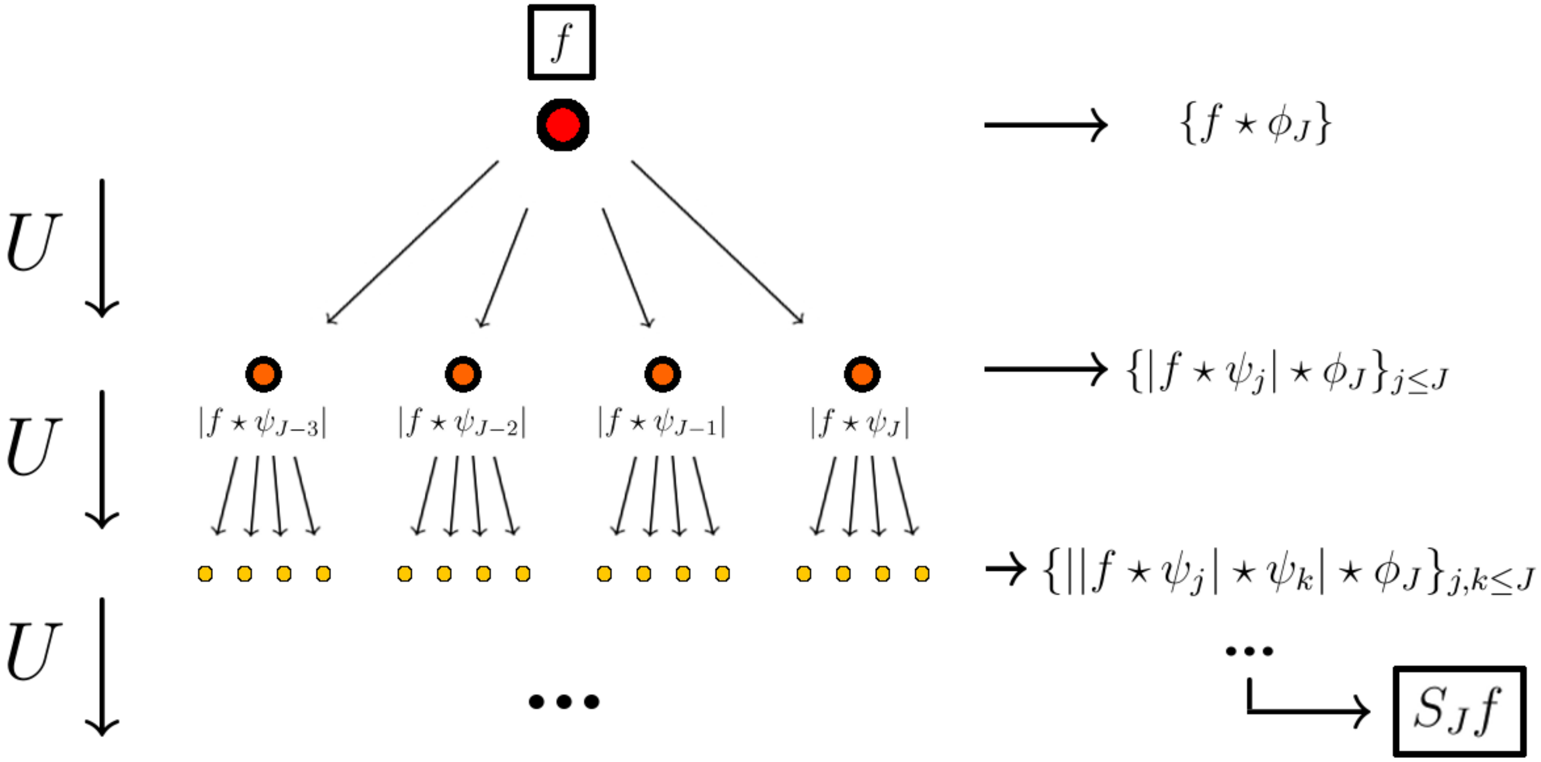




https://en.wikipedia.org/wiki/File:Typical_cnn.png

Dictionary-inspired deep architectures

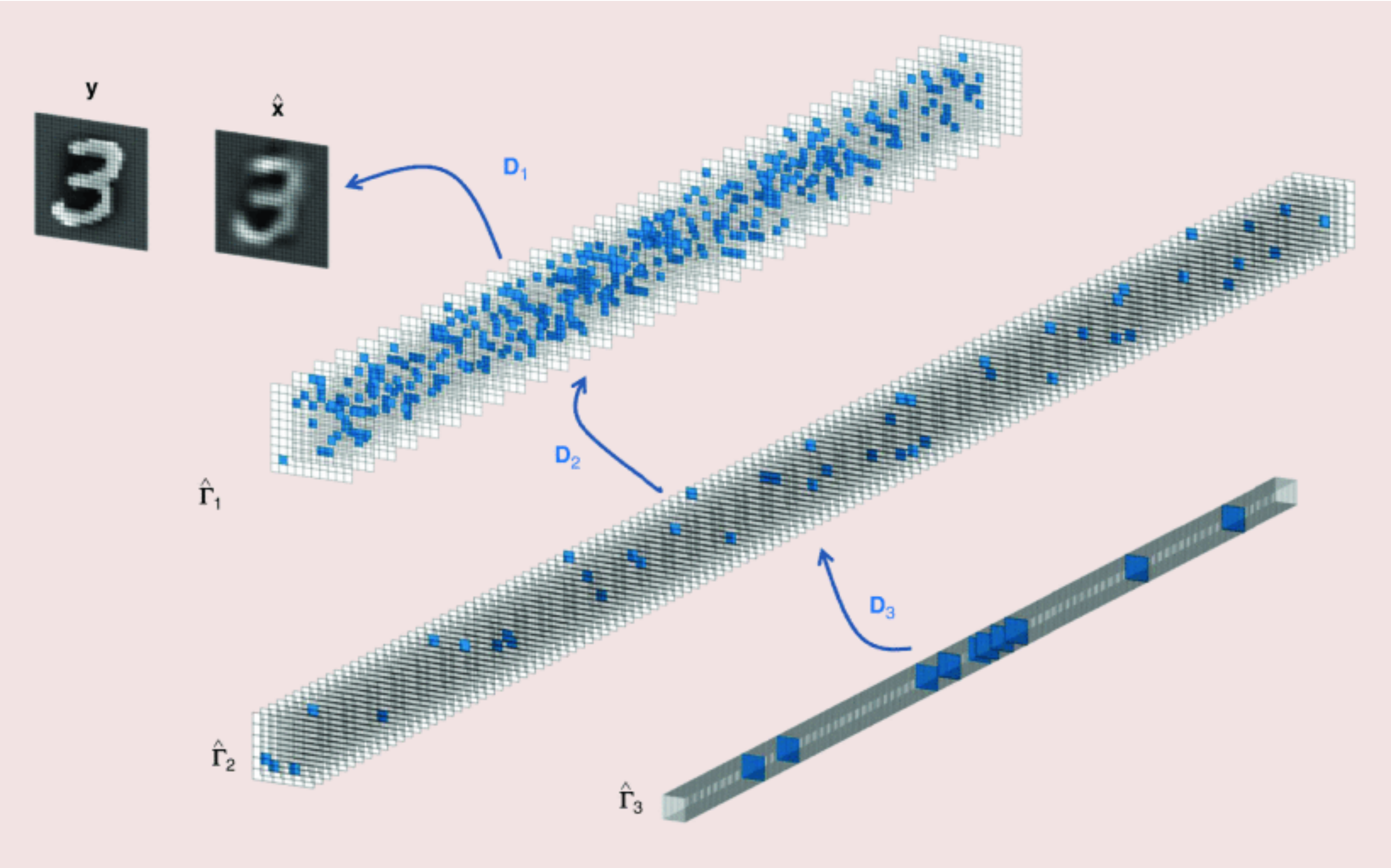
- Scattering transform



Bruna and Mallat, "Invariant scattering convolution networks," IEEE TPAMI, 2013.

Dictionary-inspired deep architectures

- Multi-layer convolutional sparse coding



Papyan et al., "Theoretical foundations of deep learning via sparse representations," IEEE SPM, 2018.

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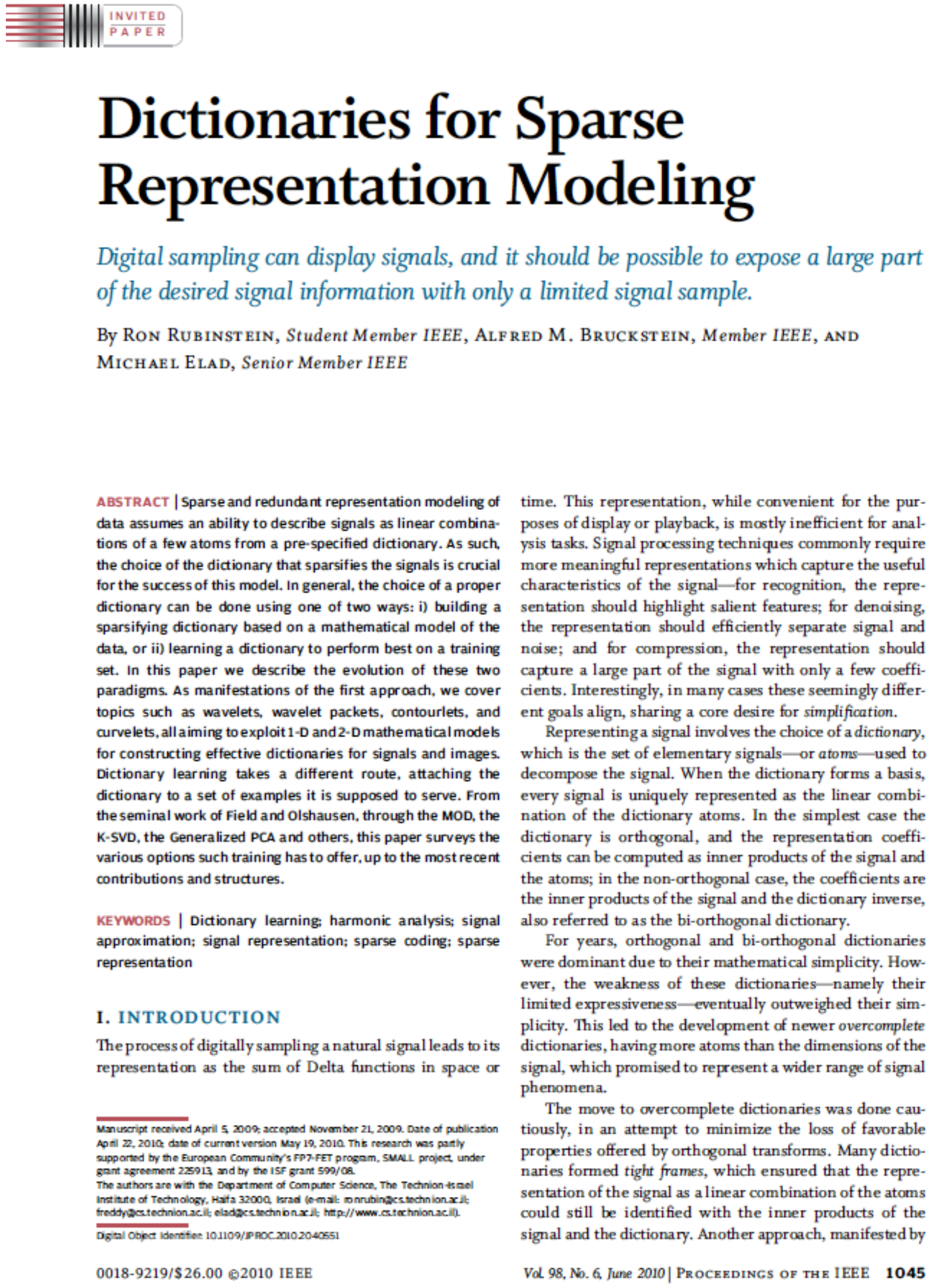

INVITED PAPER

Dictionaries for Sparse Representation Modeling

Digital sampling can display signals, and it should be possible to expose a large part of the desired signal information with only a limited signal sample.

By Ron Rubinstein, *Student Member IEEE*, Alfred M. Bruckstein, *Member IEEE*, and Michael Elad, *Senior Member IEEE*

ABSTRACT | Sparse and redundant representation modeling of data assumes an ability to describe signals as linear combinations of a few atoms from a pre-specified dictionary. As such, the choice of the dictionary that sparsifies the signals is crucial for the success of this model. In general, the choice of a proper dictionary can be done using one of two ways: i) building a sparsifying dictionary based on a mathematical model of the data, or ii) learning a dictionary to perform best on a training set. In this paper we describe the evolution of these two paradigms. As manifestations of the first approach, we cover topics such as wavelets, wavelet packets, contourlets, and curvelets, all aiming to exploit 1-D and 2-D mathematical models for constructing effective dictionaries for signals and images. Dictionary learning takes a different route, attaching the dictionary to a set of examples it is supposed to serve. From the seminal work of Field and Olshausen, through the MOD, the K-SVD, the Generalized PCA and others, this paper surveys the various options such training has to offer, up to the most recent contributions and structures.

KEYWORDS | Dictionary learning; harmonic analysis; signal approximation; signal representation; sparse coding; sparse representation

I. INTRODUCTION

The process of digitally sampling a natural signal leads to its representation as the sum of Delta functions in space or time. This representation, while convenient for the purposes of display or playback, is mostly inefficient for analysis tasks. Signal processing techniques commonly require more meaningful representations which capture the useful characteristics of the signal—for recognition, the representation should highlight salient features; for denoising, the representation should efficiently separate signal and noise; and for compression, the representation should capture a large part of the signal with only a few coefficients. Interestingly, in many cases these seemingly different goals align, sharing a core desire for *simplification*.

Representing a signal involves the choice of a *dictionary*, which is the set of elementary signals—or *atoms*—used to decompose the signal. When the dictionary forms a basis, every signal is uniquely represented as the linear combination of the dictionary atoms. In the simplest case the dictionary is orthogonal, and the representation coefficients can be computed as inner products of the signal and the atoms; in the non-orthogonal case, the coefficients are the inner products of the signal and the dictionary inverse, also referred to as the bi-orthogonal dictionary.

For years, orthogonal and bi-orthogonal dictionaries were dominant due to their mathematical simplicity. However, the weakness of these dictionaries—namely their limited expressiveness—eventually outweighed their simplicity. This led to the development of newer *overcomplete* dictionaries, having more atoms than the dimensions of the signal, which promised to represent a wider range of signal phenomena.

The move to overcomplete dictionaries was done cautiously, in an attempt to minimize the loss of favorable properties offered by orthogonal transforms. Many dictionaries formed *tight frames*, which ensured that the representation of the signal as a linear combination of the atoms could still be identified with the inner products of the signal and the dictionary. Another approach, manifested by

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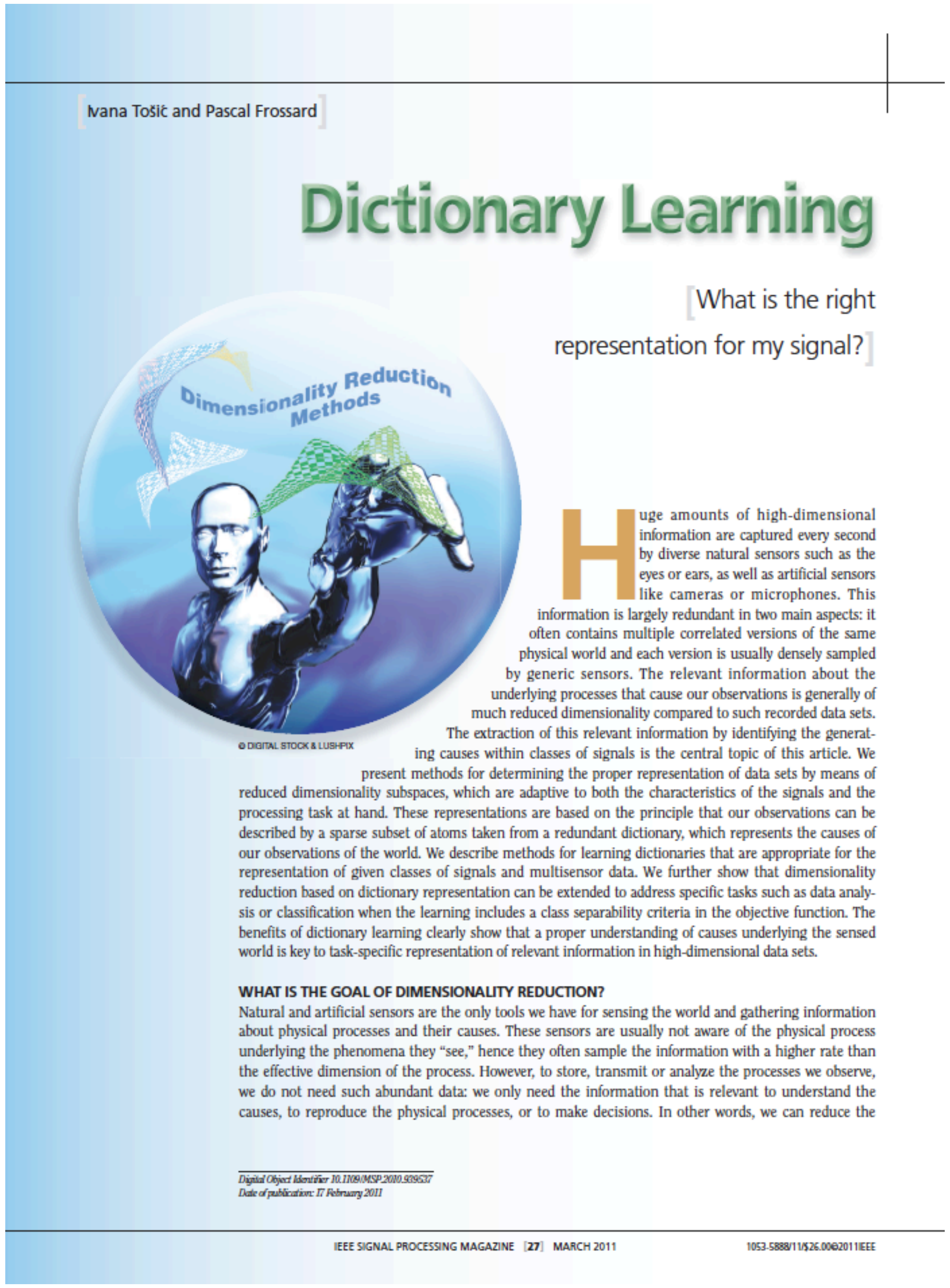

Ivana Tošić and Pascal Frossard

Dictionary Learning

[What is the right representation for my signal?]

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Huge amounts of high-dimensional information are captured every second by diverse natural sensors such as the eyes or ears, as well as artificial sensors like cameras or microphones. This information is largely redundant in two main aspects: it often contains multiple correlated versions of the same physical world and each version is usually densely sampled by generic sensors. The relevant information about the underlying processes that cause our observations is generally of much reduced dimensionality compared to such recorded data sets. The extraction of this relevant information by identifying the generating causes within classes of signals is the central topic of this article. We present methods for determining the proper representation of data sets by means of reduced dimensionality subspaces, which are adaptive to both the characteristics of the signals and the processing task at hand. These representations are based on the principle that our observations can be described by a sparse subset of atoms taken from a redundant dictionary, which represents the causes of our observations of the world. We describe methods for learning dictionaries that are appropriate for the representation of given classes of signals and multisensor data. We further show that dimensionality reduction based on dictionary representation can be extended to address specific tasks such as data analysis or classification when the learning includes a class separability criteria in the objective function. The benefits of dictionary learning clearly show that a proper understanding of causes underlying the sensed world is key to task-specific representation of relevant information in high-dimensional data sets.

WHAT IS THE GOAL OF DIMENSIONALITY REDUCTION?

Natural and artificial sensors are the only tools we have for sensing the world and gathering information about physical processes and their causes. These sensors are usually not aware of the physical process underlying the phenomena they "see," hence they often sample the information with a higher rate than the effective dimension of the process. However, to store, transmit or analyze the processes we observe, we do not need such abundant data: we only need the information that is relevant to understand the causes, to reproduce the physical processes, or to make decisions. In other words, we can reduce the

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