AIMS CDT - Signal Processing
Michaelmas Term 2022

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Outline

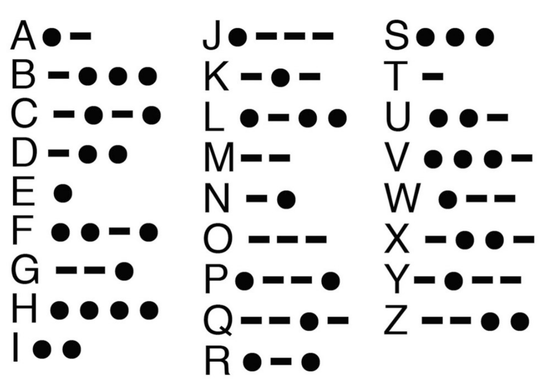
- Lectures
  - Monday-Thursday 10:00-12:00
  - lecture slides: http://www.robots.ox.ac.uk/~xdong/teaching.html
- Lab sessions
  - Tuesday-Thursday 14:00-17:00
  - lab notes: http://www.robots.ox.ac.uk/~xdong/teaching.html
  - lab demonstrators
    - Yin-Cong Zhi (yin-cong.zhi@st-annes.ox.ac.uk)
    - Pierre Osselin (pierre.osselin@eng.ox.ac.uk)
    - Bohan Tang (bohan.tang@eng.ox.ac.uk)
  - light-weight assessment (exercise at the end of Lab 2)
    - to be submitted to xdong@robots.ox.ac.uk by Monday Oct 24th 18:00
- Questions & Comments
  - Xiaowen Dong (xdong@robots.ox.ac.uk)

Outline

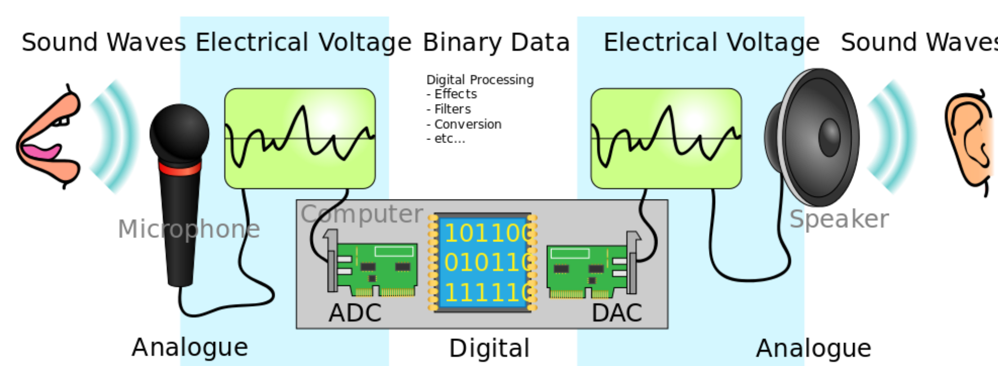
- Part I Classical signal processing
  - Day 1: Basic concepts and tools (thanks to Steve Roberts)
    - linear systems, convolution, time-frequency analysis, filtering, analogue & digital filters, discrete Fourier transform

smoke signal (1570)

semaphore telegraph (1792)



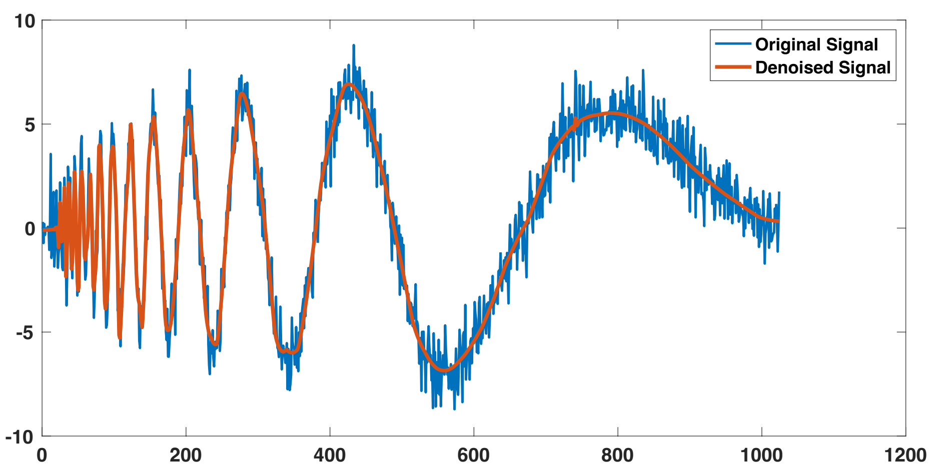
Morse code (1830s)



speech processing (1950s-)

Outline

- Part I Classical signal processing
  - Day 2: Representation of signals
    - stochastic models, time-frequency representation, transforms, dictionary learning
    - Lab 1: Autoregressive models



signal denoising

original **JPEG 2000 (1% in size)**

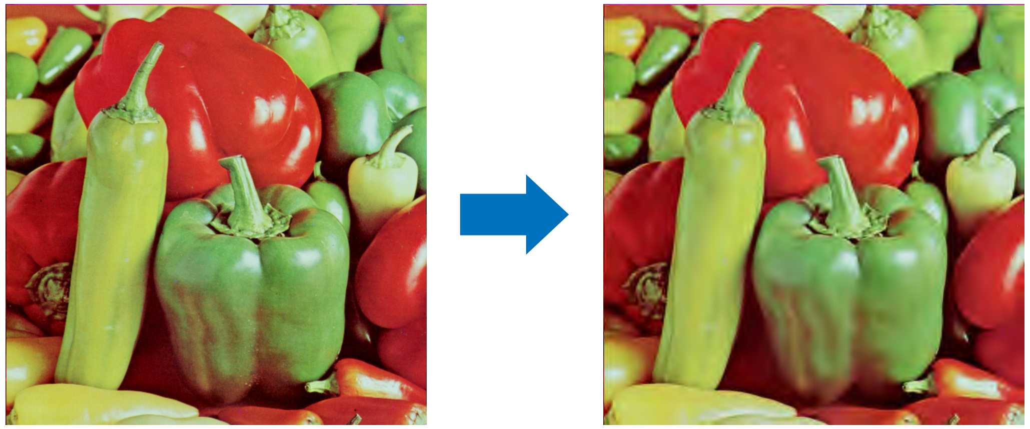
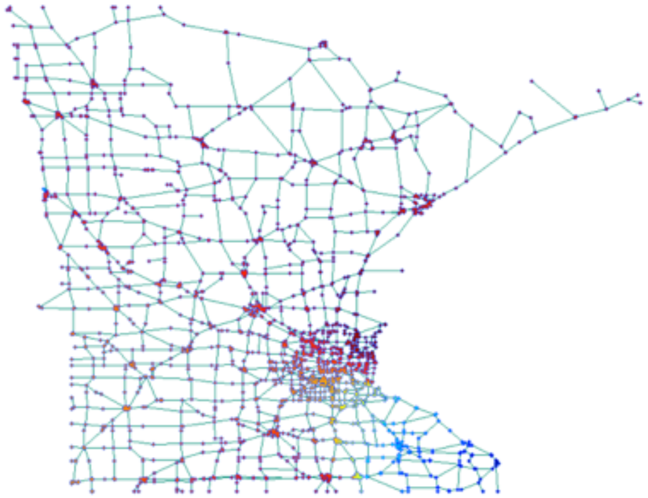


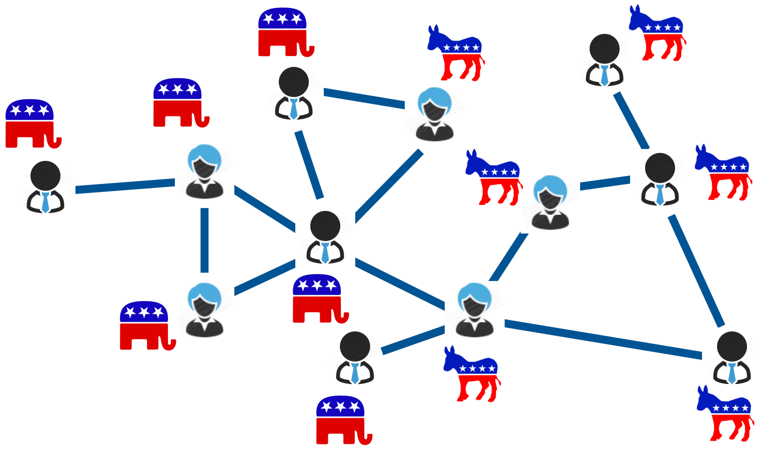
image compression

Outline

- Part II Graph signal processing
  - Day 3: Introduction to graph signal processing
    - graph signals, graph Fourier transform, filtering, representation of graph signals
    - Lab 2: Graph signal processing



traffic congestion

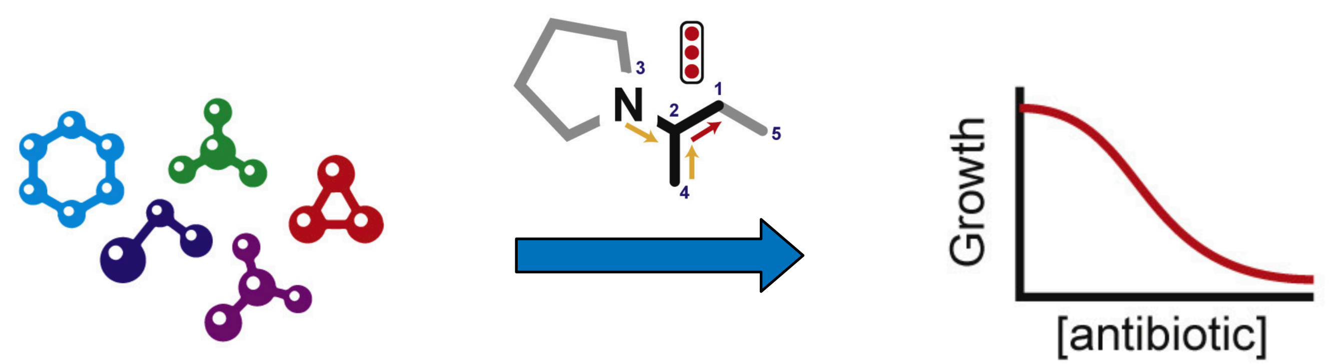


political preference

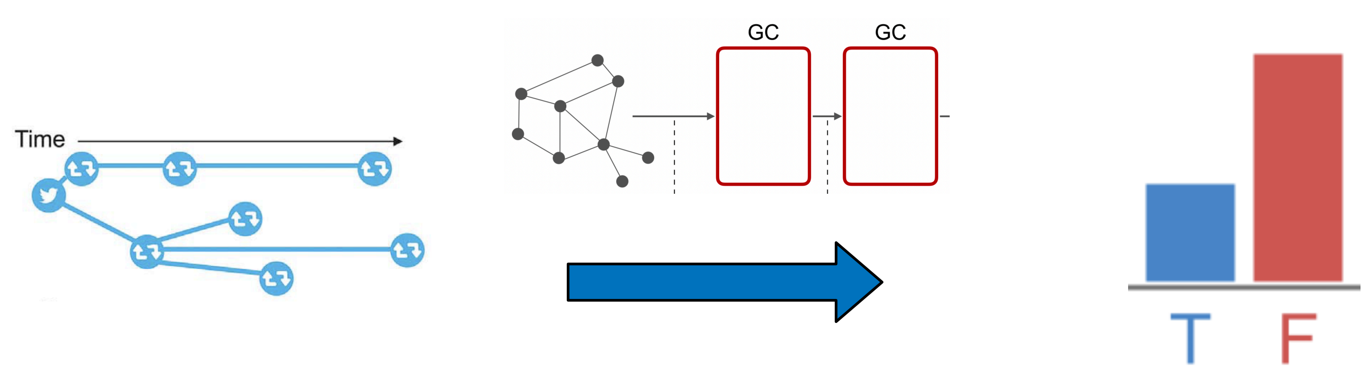
Outline

- Part II Graph signal processing
  - Day 4: Introduction to graph machine learning (by Dorina Thanou)
    - graph ML tasks, graph convolution, spatial-domain vs spectral-domain approaches
    - Lab 3: Graph neural networks

drug discovery

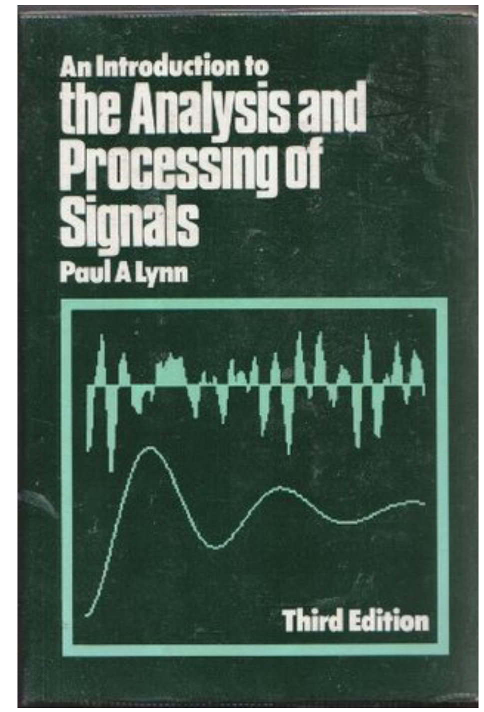


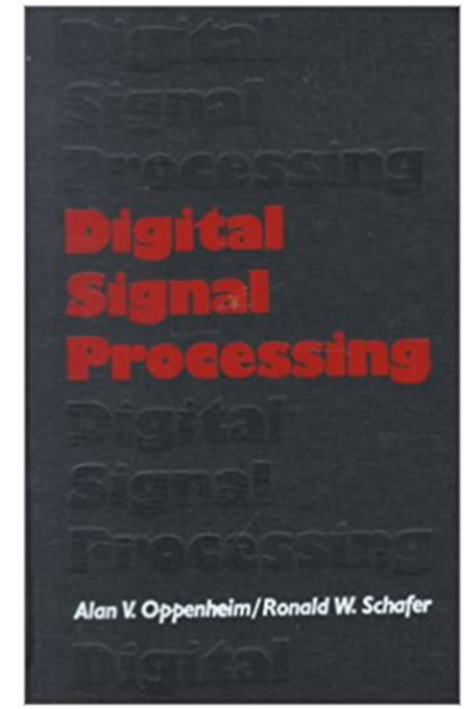
fake news detection

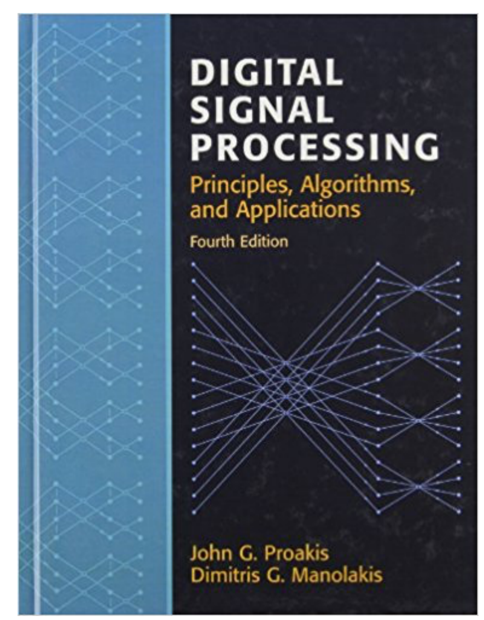


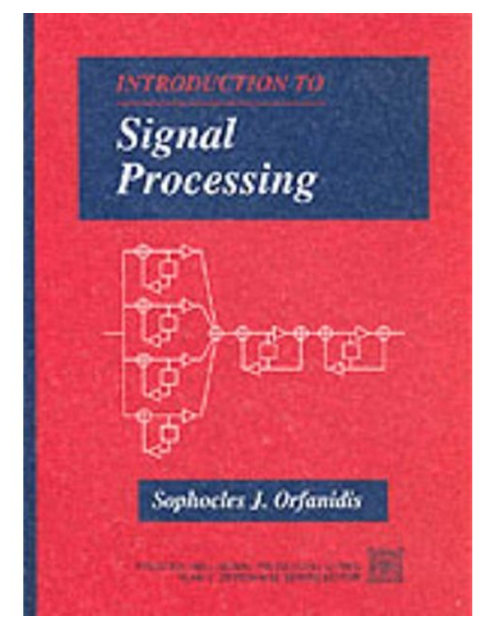
Textbooks (Part I)

- Lynn. An introduction to the analysis and processing of signals. Macmillan, 1989.
- Oppenheim and Schafer. Digital signal processing. Prentice Hall, 1975.
- Proakis and Manolakis. Digital signal processing: Principles, algorithms and applications. Prentice Hall, 2007
- Orfanidis. Introduction to signal processing. Prentice Hall, 1996. Available online at http://eceweb1.rutgers.edu/~orfanidi/intro2sp/



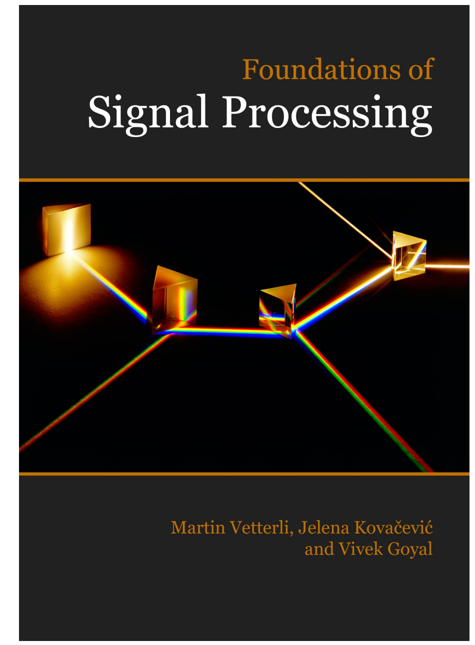


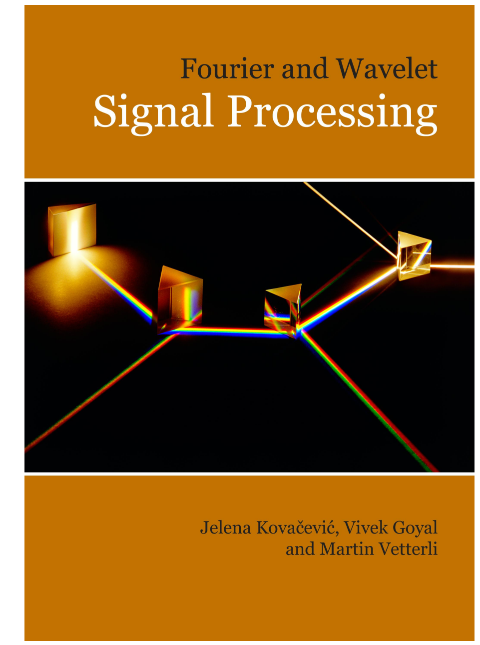




Textbooks (Part I)

- Vetterli et al. Foundations of signal processing. Cambridge University Press, 2014. Available online at http://www.fourierandwavelets.org
- Kovačević et al. Fourier and wavelet signal processing. Available online at http://www.fourierandwavelets.org



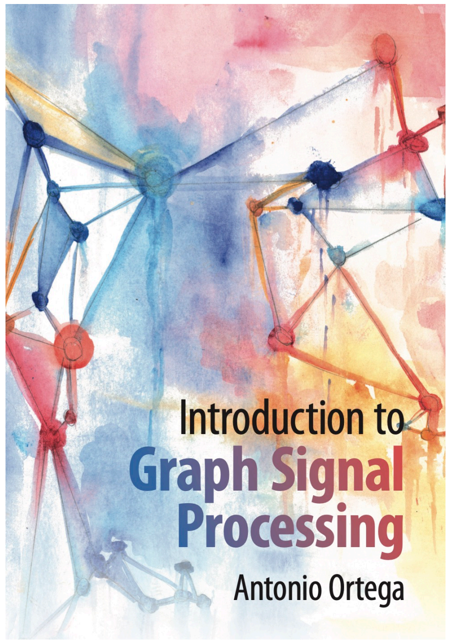


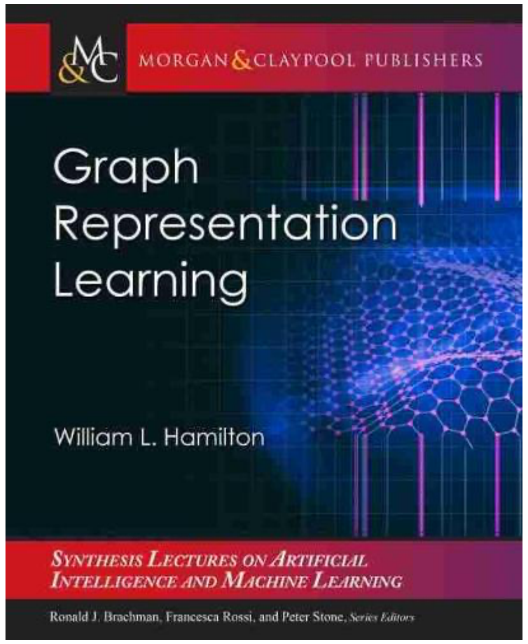
Toolboxes

- MATLAB Signal Processing Toolbox:
  - https://www.mathworks.com/help/signal/
- SciPy Signal Processing Toolbox:
  - https://docs.scipy.org/doc/scipy/reference/tutorial/signal.html
  - https://scipy-cookbook.readthedocs.io/items/idx_signal_processing.html

Textbooks (Part II)

- Ortega. Introduction to graph signal processing. Cambridge University Press, 2022.
- Hamilton. Graph representation learning. Morgan & Claypool Publishers, 2020. Available online at https://www.cs.mcgill.ca/~wlh/grl_book/



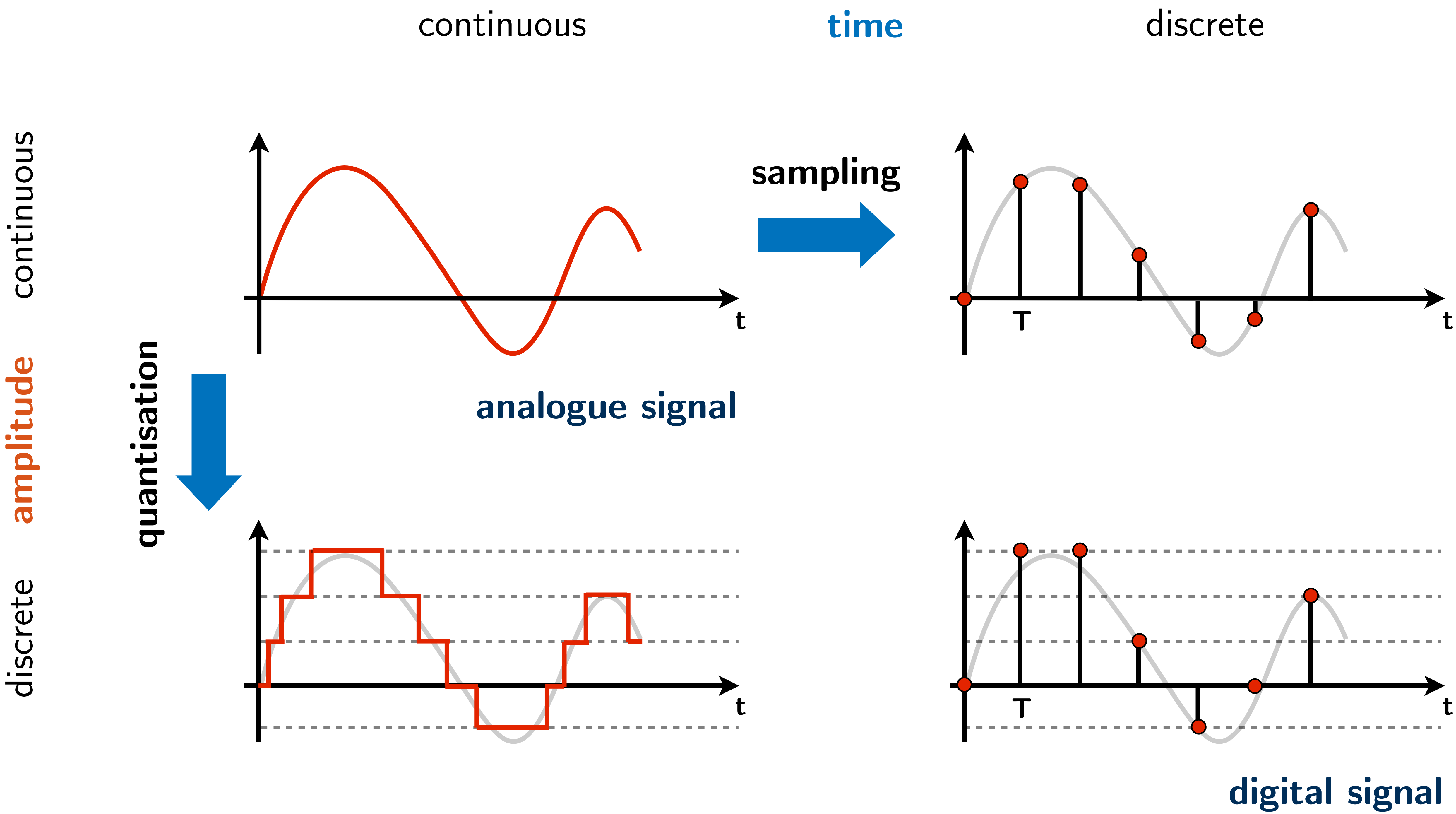


Resources

- https://web.media.mit.edu/~xdong/resource.html
- https://github.com/naganandy/graph-based-deep-learning-literature

Basic Concepts and Tools

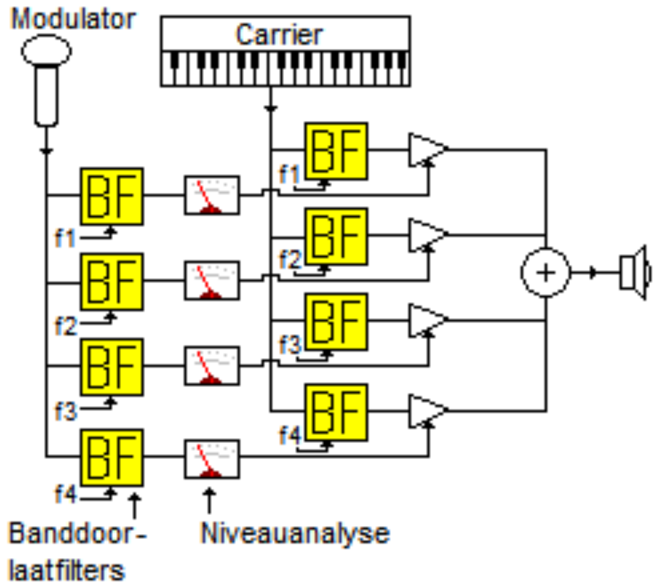
Signal types



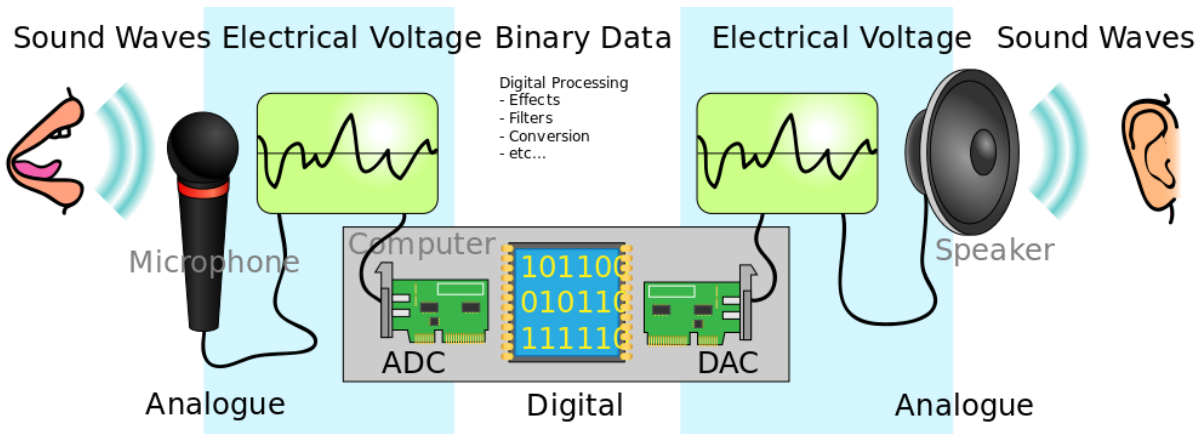
Analogue vs. Digital signal processing

- Many signals of practical interest are analogue: e.g., speech, seismic, radar, and sonar signals

- Analogue signal processing systems are based on analogue equipment: e.g., channel vocoder



- Dramatic advance of digital computing moves the trend towards digital systems

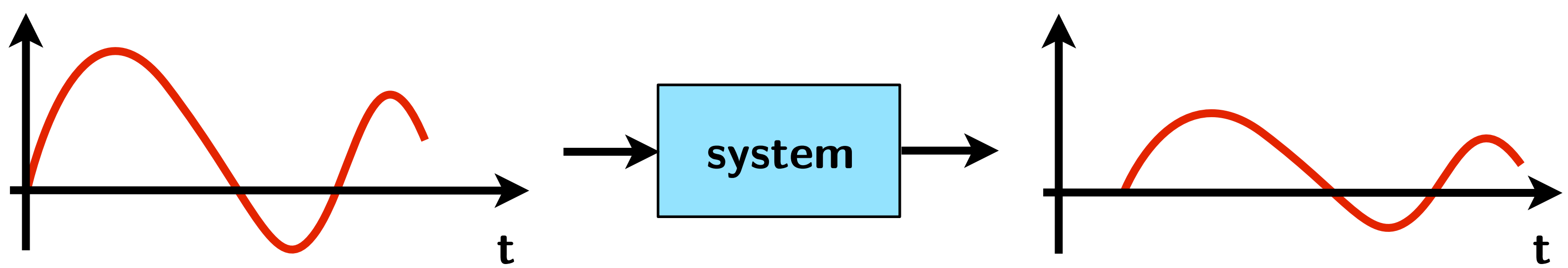


Linear systems

- Principle of superposition

$$ax_1(t) + bx_2(t) \longrightarrow \boxed{\textbf{system}} \longrightarrow ay_1(t) + by_2(t)$$

- Frequency preservation: $\mathcal{F}_{\text{out}} \subseteq \mathcal{F}_{\text{in}}$



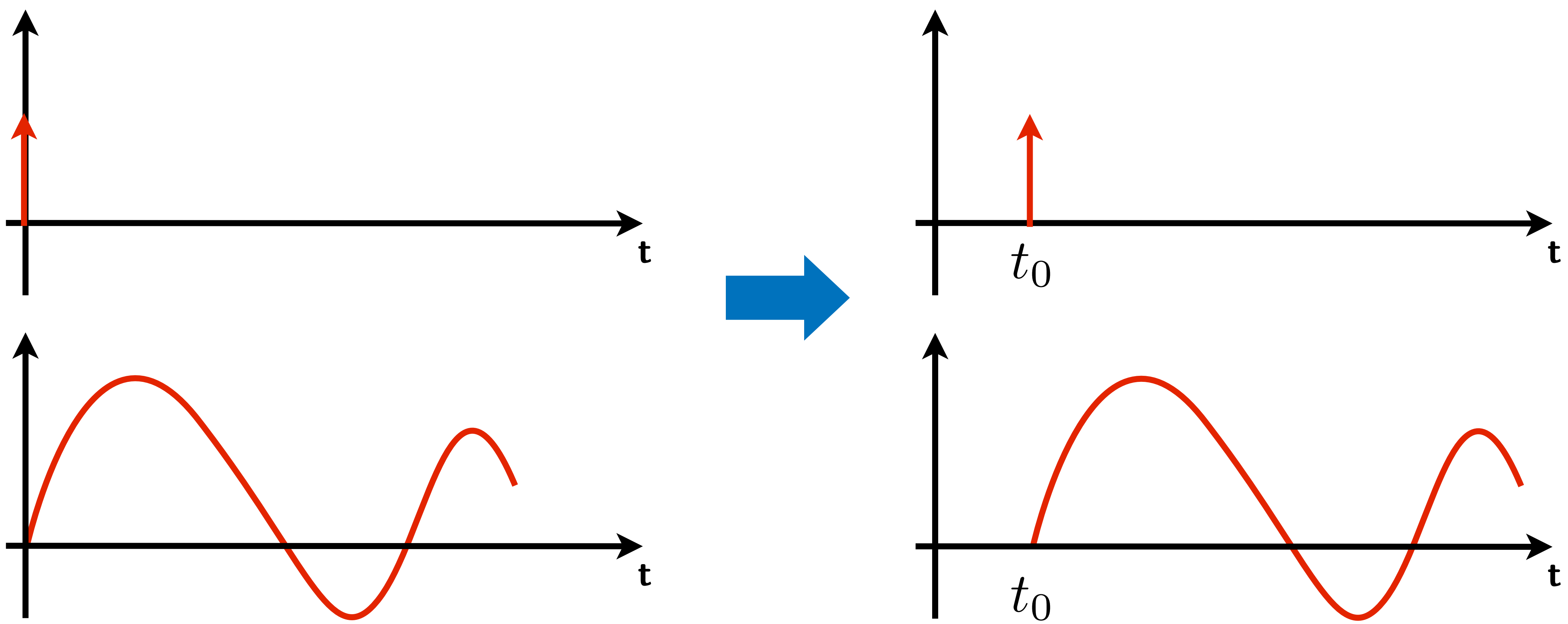
- Can be broken down into simpler sub-systems

$$x \longrightarrow \boxed{\textbf{system}} \quad \equiv \quad x \longrightarrow \boxed{\textbf{sub-system}} \longrightarrow \boxed{\textbf{sub-system}} \quad \equiv \quad x \longrightarrow \boxed{\textbf{sub-system}} \longrightarrow \boxed{\textbf{sub-system}}$$

Time-invariant systems

- Time-invariance

$$x(t) \longrightarrow \boxed{\textbf{system}} \longrightarrow y(t) \quad \Longrightarrow \quad x(t-t_0) \longrightarrow \boxed{\textbf{system}} \longrightarrow y(t-t_0)$$



Linear time-invariant (LTI) systems

- Linear time-invariant (LTI) systems are both linear and time-invariant

$$y(t) = [x(t)]^2 \ ☹ \quad y(t) = x(2t) \ ☹ \quad y(t) = x(t) - x(t-1) \ ☺$$

- Causality: "present" only depends on "present" and "past"

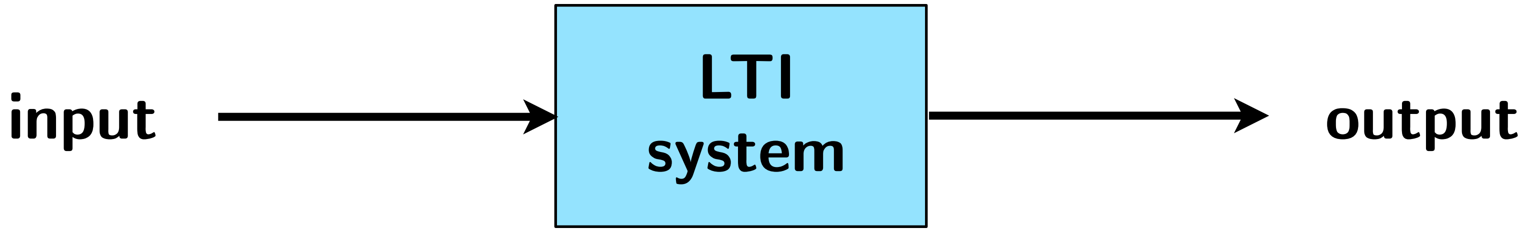
$$y(t) = x(t+1) - x(t) \ ☹ \qquad y(t) = x(t) - x(t-1) \ ☺$$

- Stability: a system is bounded-input bounded-output (BIBO) stable if

$$|x(t)| \leq M_x < \infty \quad \Rightarrow \quad |y(t)| \leq M_y < \infty$$

$$y(t) = \frac{1}{x(t)} \ ☹ \qquad y(t) = x(t) - x(t-1) \ ☺$$

Linear processes



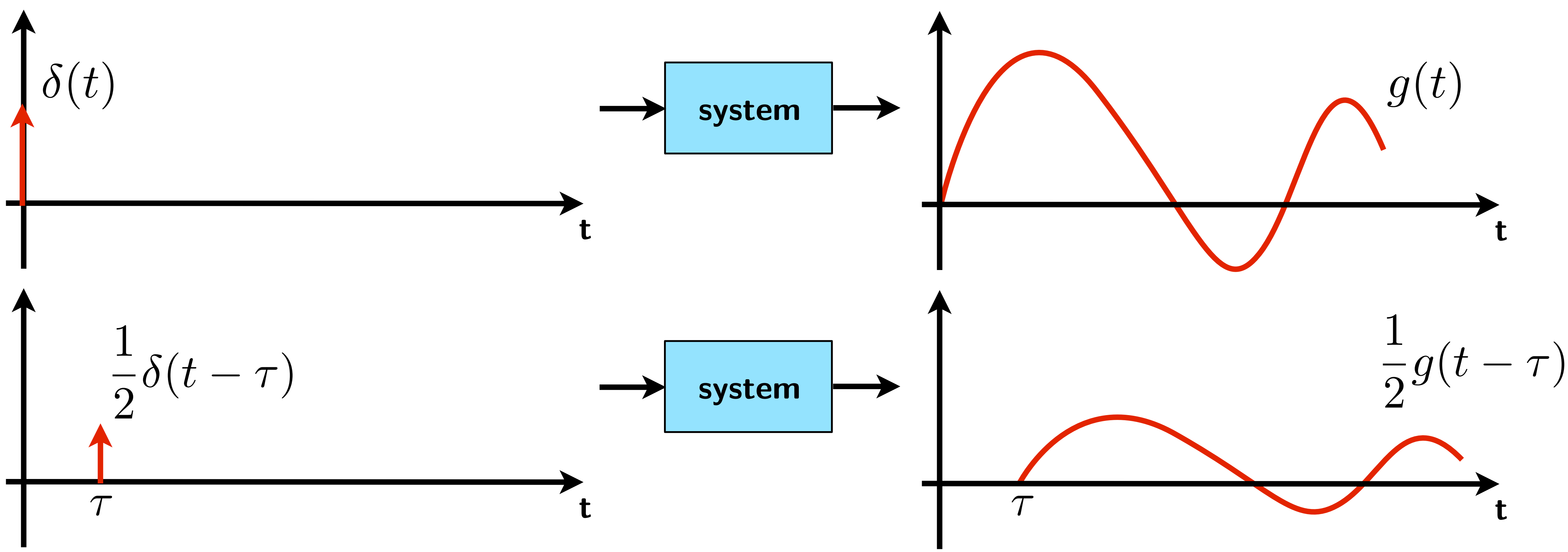
amplification, (un)-mixing, filtering, etc.

- Input-output characteristics can be defined by
  - **impulse response** in the **time** domain
  - **transfer function** in the **frequency** domain

- There is an **invertible** mapping between time- and frequency-domain representations

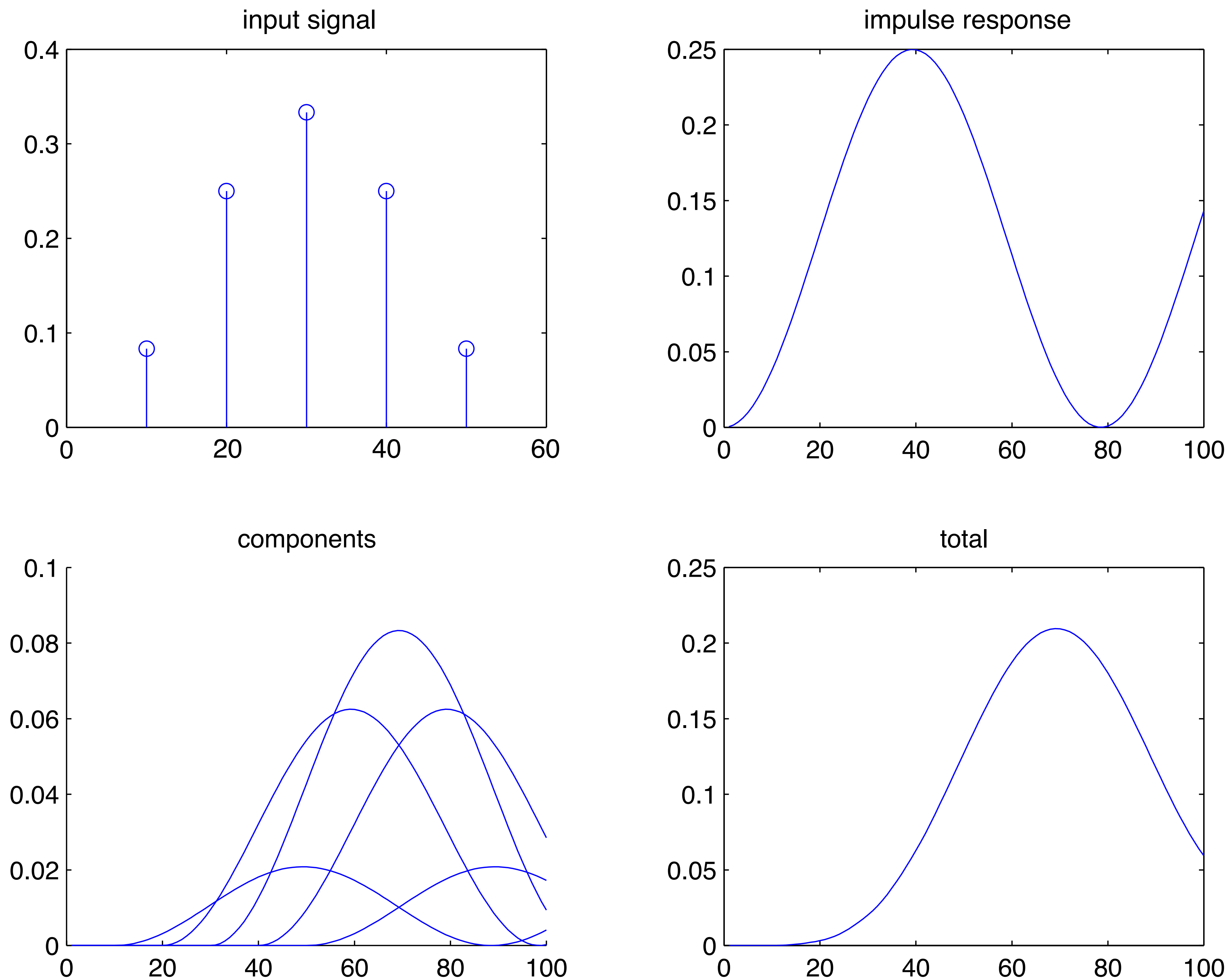
Time-domain analysis - Convolution

- Convolution allows the evaluation of the output signal from an LTI system, given its **impulse response** and input signal

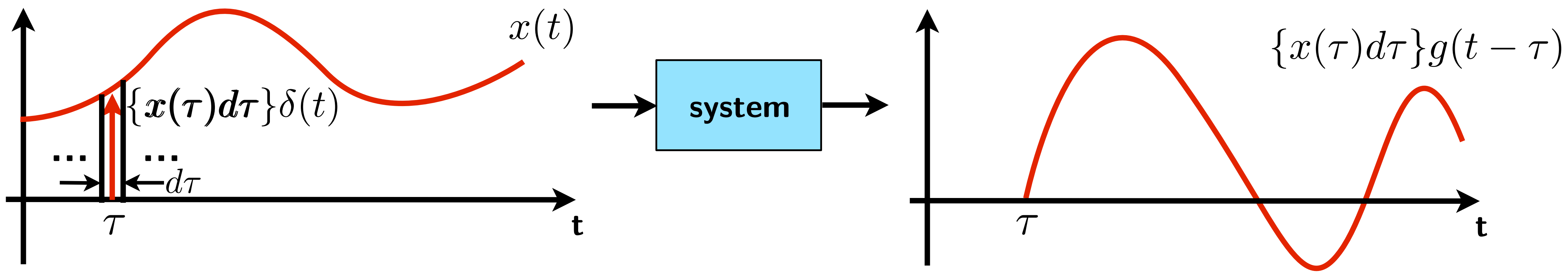


- Evaluate system output for
  - input: succession of impulse functions (which generate weighted impulse responses)
  - output: sum of the effect of each impulse function

Time-domain analysis - Convolution



Time-domain analysis - Convolution



- this gives the **convolution integral**

$$y(t) = \sum_{\tau}\{x(\tau)d\tau\}g(t-\tau) \xrightarrow{d\tau \to 0} \int_0^{\infty} x(\tau)g(t-\tau)d\tau$$

- the system response is the **convolution** of the input and the impulse response
- the system is **completely** characterised by impulse response in time-domain

Time-domain analysis - Convolution

- Convolution is commutative

$$y(t) = \int_0^\infty x(\tau)g(t-\tau)d\tau = \int_0^\infty x(t-\tau)g(\tau)d\tau$$

- Convolution vs. Correlation

$$f(t) = \int_{-\infty}^\infty x(\tau)g(t-\tau)d\tau$$ integral over **lags** at a fixed **time**

$$R_{xy}(\tau) = \int_{-\infty}^\infty x(t)y(t-\tau)dt$$ integral over **time** for a fixed **lag**

Frequency-domain analysis

- Consider the following LTI system

$$x(t) = e^{st} \longrightarrow \boxed{g(t)} \longrightarrow y(t)$$

$$y(t) = \int_{-\infty}^{\infty} e^{s\tau} g(t-\tau)d\tau = \underbrace{\boxed{\int_{-\infty}^{\infty} g(t)e^{-st}dt}}_{G(s)} \cdot e^{st}$$

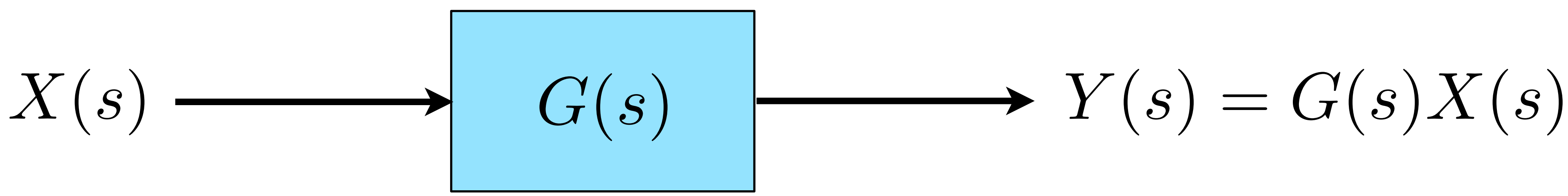
  - e^{st} is an **eigenfunction** of an LTI system with eigenvalue $G(s)$, which is the Laplace transform of the impulse response $g(t)$
  - knowledge of $G(s)$ for all s **completely** characterises the system

The Laplace transform

- Laplace transform of $x(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

- Transfer function $G(s)$



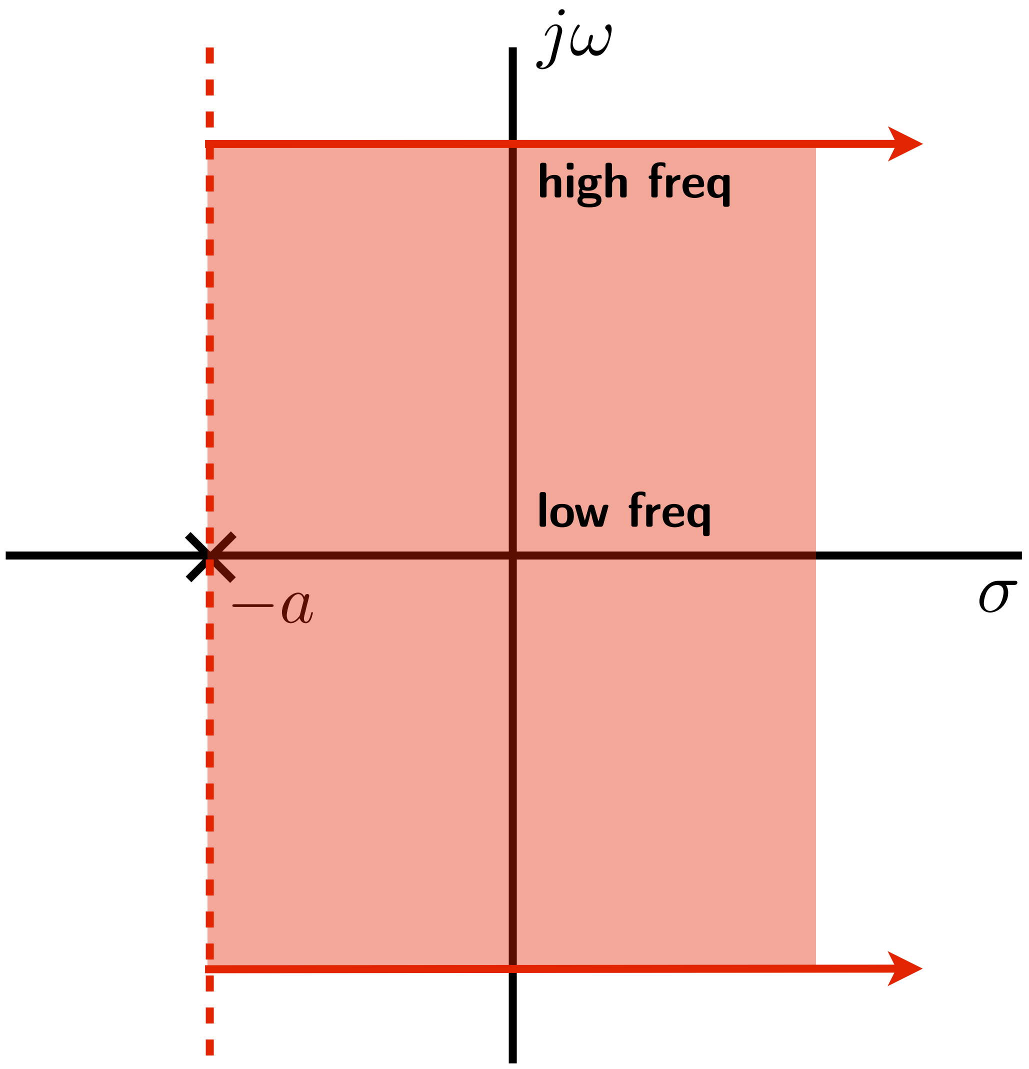
- Can be expressed as a **pole-zero** representation of the form

$$G(s) = \frac{A(s - z_1)\dots(s - z_m)}{(s - p_1)(s - p_2)\dots(s - p_n)}$$

pole at infinity ($G(\infty) = \infty$) if $n < m$
zero at infinity ($G(\infty) = 0$) if $n > m$

The Laplace transform and LTI system

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}dt = \int_{-\infty}^{\infty} g(t)e^{-\sigma t}e^{-j\omega t}dt < \infty$$



s-plane

$g(t) = e^{-at}u(t)$ where $a > 0$

region of convergence (ROC): $\sigma > -a$

$$G(s) = \frac{1}{s+a}$$

- **causal system:** if the ROC extends rightward from the rightmost pole and $n \geq m$
- **stable system:** ROC includes the imaginary axis
- **causal and stable system:** all poles must be in the left-half of the s-plane

The Fourier transform

- Laplace transform reduces to Fourier transform with $s = j\omega$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

- Transfer function reduces to frequency response $G(j\omega)$

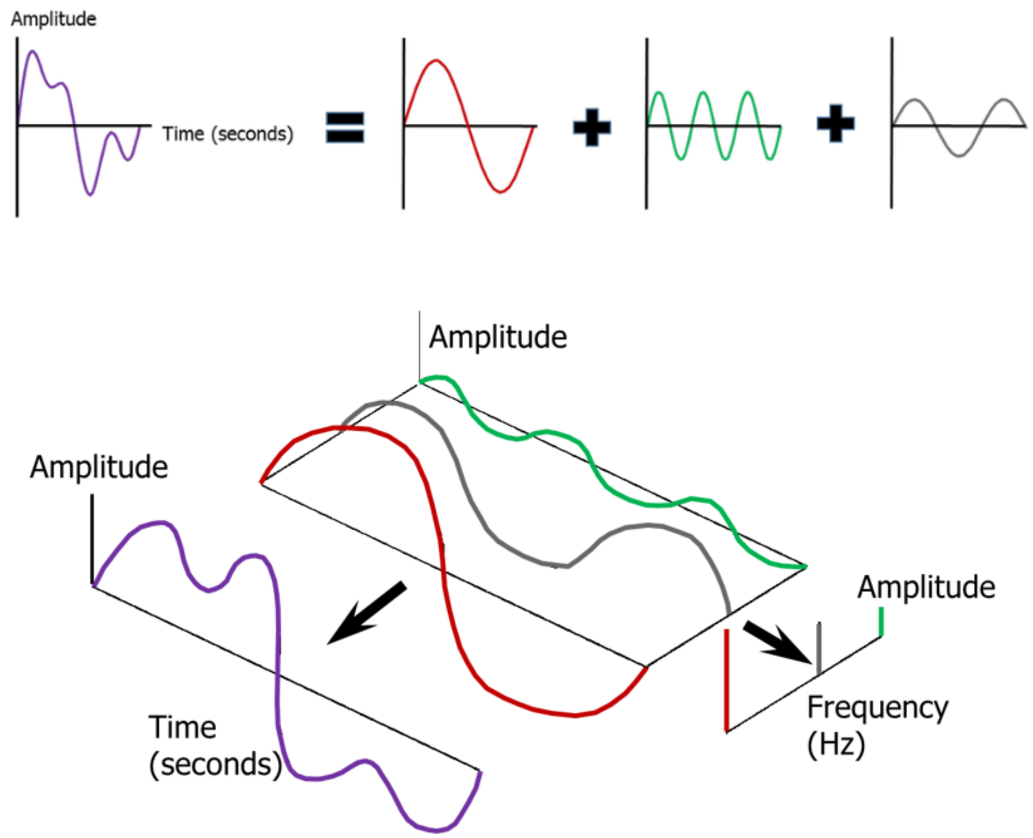
$$X(j\omega) \longrightarrow \boxed{G(j\omega)} \longrightarrow Y(j\omega) = G(j\omega)X(j\omega)$$

- Inverse Fourier transform

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\omega) e^{j\omega t} d\omega$$

The Fourier series

- **Fourier series** for **periodic signal**



fundamental frequency $f_0 = 1/T_0$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_0 t}$$

$$C_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j2\pi n f_0 t} dt$$

- When the period approaches infinity, the spectrum becomes continuous leading to **Fourier transform** for **aperiodic signal** (previous slide)

Laplace vs. Fourier transform

Laplace transform	**Fourier transform**
$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$	$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
complex s-plane	imaginary axis of complex s-plane
transfer function	frequency response
may exist when FT doesn't	may exist when LT doesn't

Time-domain vs. Frequency-domain

- **Theorem**

  If $g(t)$ is the **impulse response** of an LTI system, then its Fourier transform, $G(j\omega)$, is the **frequency response** of the system

- **Proof** Consider $x(t) = A\cos\omega t$, by convolution:

$$
\begin{aligned}
y(t) &= \int_0^{\infty} A\cos\omega(t-\tau)g(\tau)d\tau \\
&= \frac{A}{2}\int_0^{\infty} e^{j\omega(t-\tau)}g(\tau)d\tau + \frac{A}{2}\int_0^{\infty} e^{-j\omega(t-\tau)}g(\tau)d\tau \\
&= \frac{A}{2}e^{j\omega t}\int_{-\infty}^{\infty} g(\tau)e^{-j\omega\tau}d\tau + \frac{A}{2}e^{-j\omega t}\int_{-\infty}^{\infty} g(\tau)e^{j\omega\tau}d\tau \\
&= \frac{A}{2}\{e^{j\omega t}G(j\omega) + e^{-j\omega t}G(-j\omega)\}
\end{aligned}
$$

Time-domain vs. Frequency-domain

Let $G(j\omega) = Ce^{j\phi}$, i.e., $C = |G(j\omega)|, \;\; \phi = arg\{G(j\omega)\}$

then $y(t) = \dfrac{AC}{2}\{e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}\} = CA\cos(\omega t + \phi)$

that is, an input sinusoid has its amplitude scaled by $|G(j\omega)|$ and phase changed by $arg\{G(j\omega)\}$, where $G(j\omega)$ is the Fourier transform of the impulse response $g(t)$.

Time-domain vs. Frequency-domain

- **Theorem**

  Convolution in the time domain is equivalent to multiplication in the frequency domain, i.e.,

$$y(t) = g(t) * x(t) \equiv \mathcal{F}^{-1}\{Y(j\omega) = G(j\omega)X(j\omega)\}$$

$$y(t) = g(t) * x(t) \equiv \mathcal{L}^{-1}\{Y(s) = G(s)X(s)\}$$

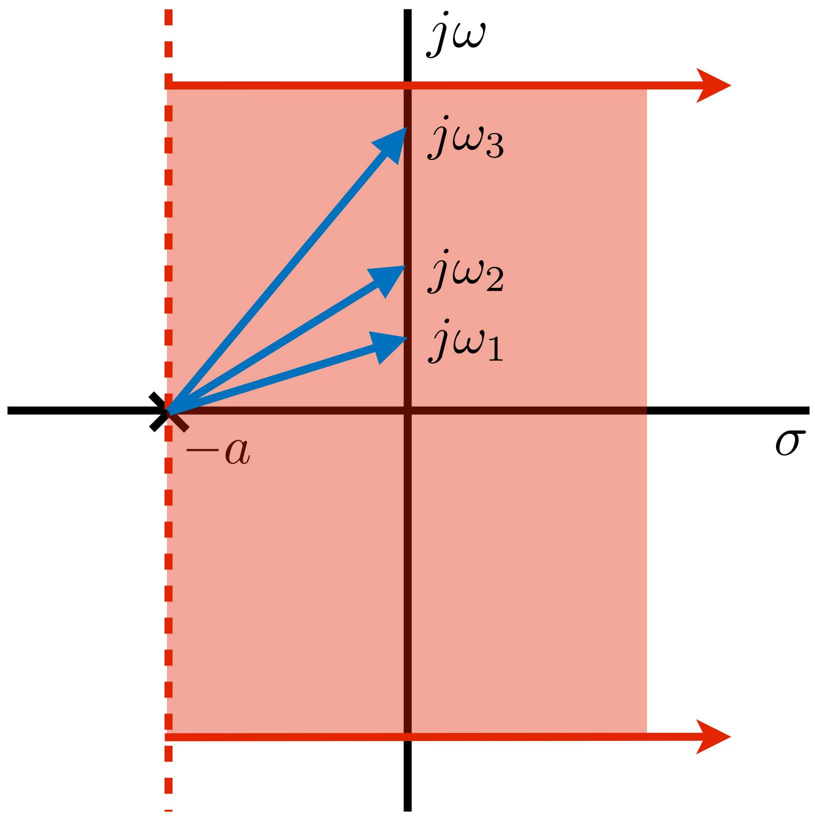
- **Proof**

$$\begin{aligned}\mathcal{L}\{f(t) * g(t)\} &= \int_t \int_\tau f(t-\tau) g(\tau) d\tau e^{-st} dt \\ &= \int_\tau g(\tau) e^{-s\tau} d\tau \mathcal{L}\{f(t)\} \\ &= \mathcal{L}\{g(t)\}\mathcal{L}\{f(t)\}\end{aligned}$$

By letting $s = j\omega$ we prove the result for the Fourier transform.

Time-domain vs. Frequency-domain

- We can move losslessly between **time** and **frequency** domains, choosing whichever is the easier to work with
- Convolution theorem provides the mathematical underpinning that helps understand stability and characteristics of linear systems such as filters

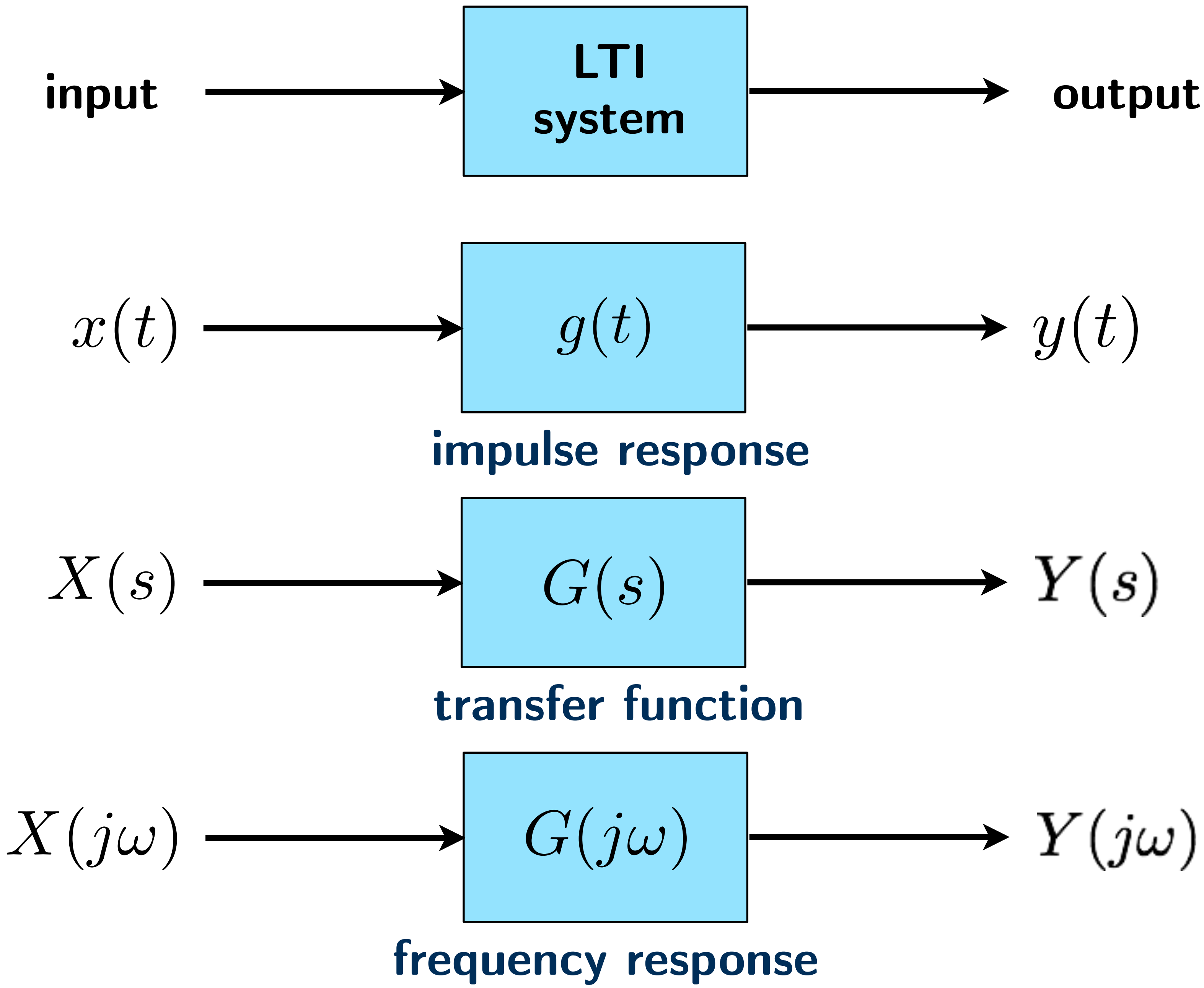


s-plane

- **stable system:** ROC extends rightward from the rightmost pole and $n \geq m$
- **low-pass system:** frequency response can be analysed by drawing vectors from poles and zeros to imaginary axis

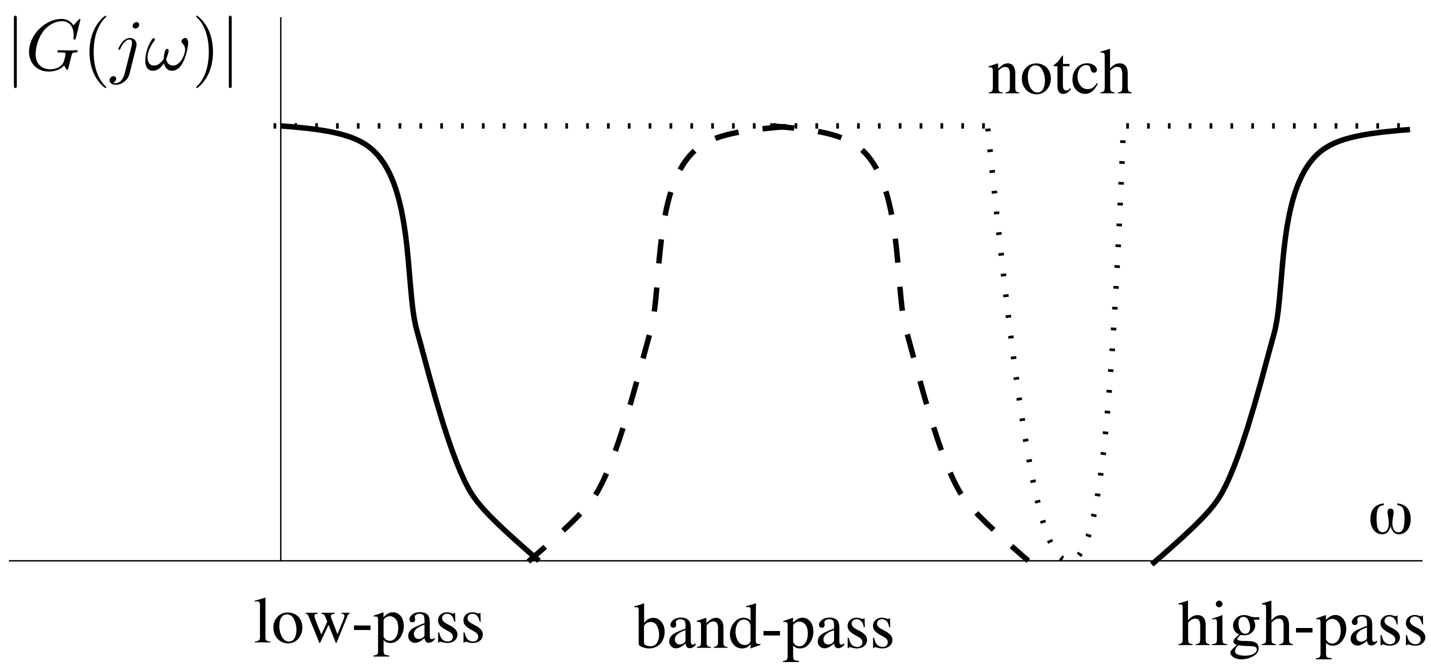
Filtering

- Filtering as input-output relationship



Filtering

- Filters are **frequency-selective** linear systems
  - **Low-pass**: extract short-term average or to eliminate high-frequency fluctuations
  - **High-pass**: follow small-amplitude high-frequency perturbations in presence of much larger slowly-varying component
  - **Band-pass**: select a required modulated carrier frequency out of many
  - **Band-stop**: eliminate single-frequency interference (also known as notch filtering)



Design of analogue filters

- A filter may be described by its impulse response or by its frequency response (or transfer function)

- Filter design takes into account
  - the desired magnitude response
  - the desired phase response

- Squared magnitude of the transfer function

$$|G(s)|^2 = G(s)G^*(s) \overset{s = j\omega}{=\!=\!=} G(s)G(-s)$$

- Design procedure
  - consider some desired response $|G(s)|^2$ as a ratio of two polynomials in even powers of s (or ω)
  - design the filter by assigning "stable" poles (those on L.H.S of s-plane) to $G(s)$

Butterworth low-pass filters

$$|G(j\omega)|^2 = \frac{1}{1 + H\{(\frac{\omega}{\omega_c})^2\}} = \frac{1}{1 + (\frac{\omega}{\omega_c})^{2n}}$$

n : order of the filter

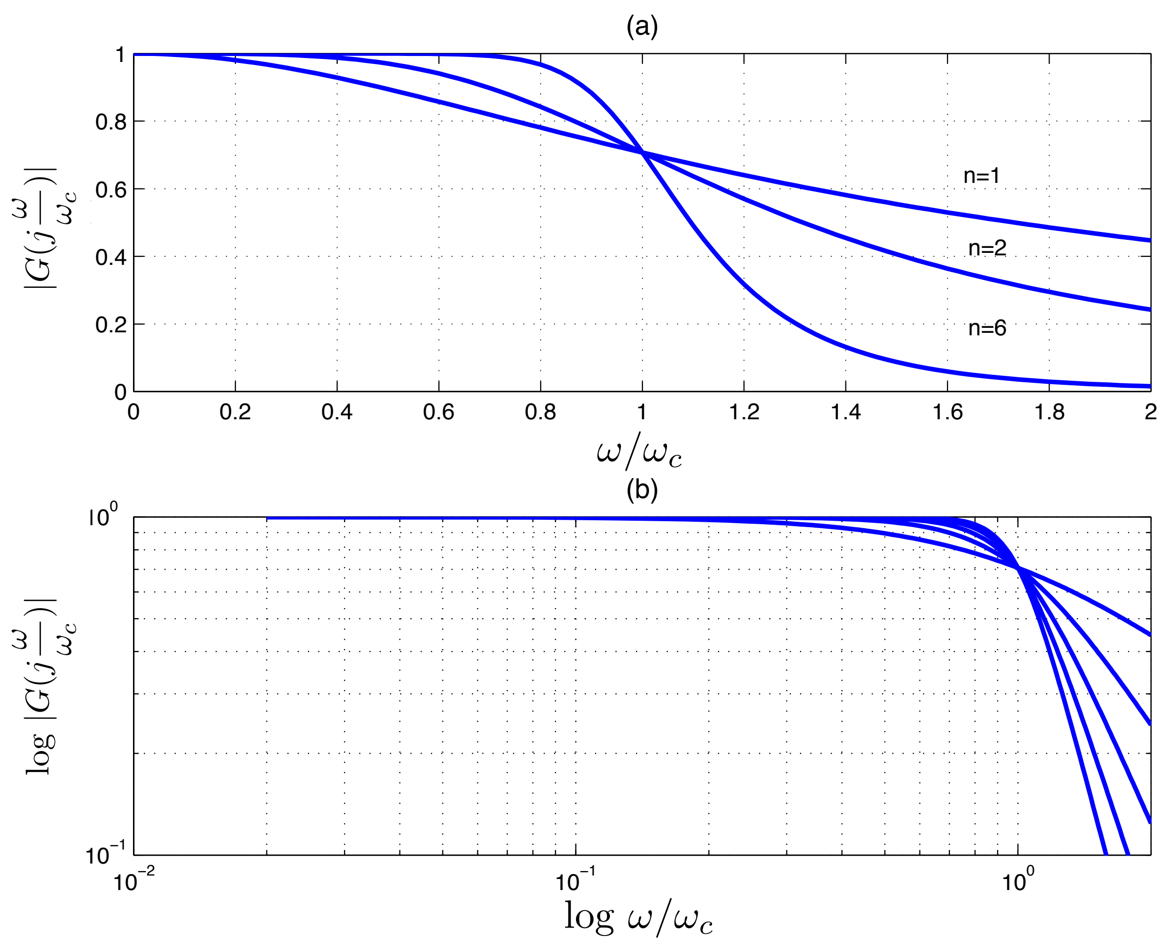


Figure 1.5: *Butterworth filter response on (a) linear and (b) log scales. On a log-log scale the response, for $\omega > \omega_c$ falls off at approx -20db/decade.*

frequency response

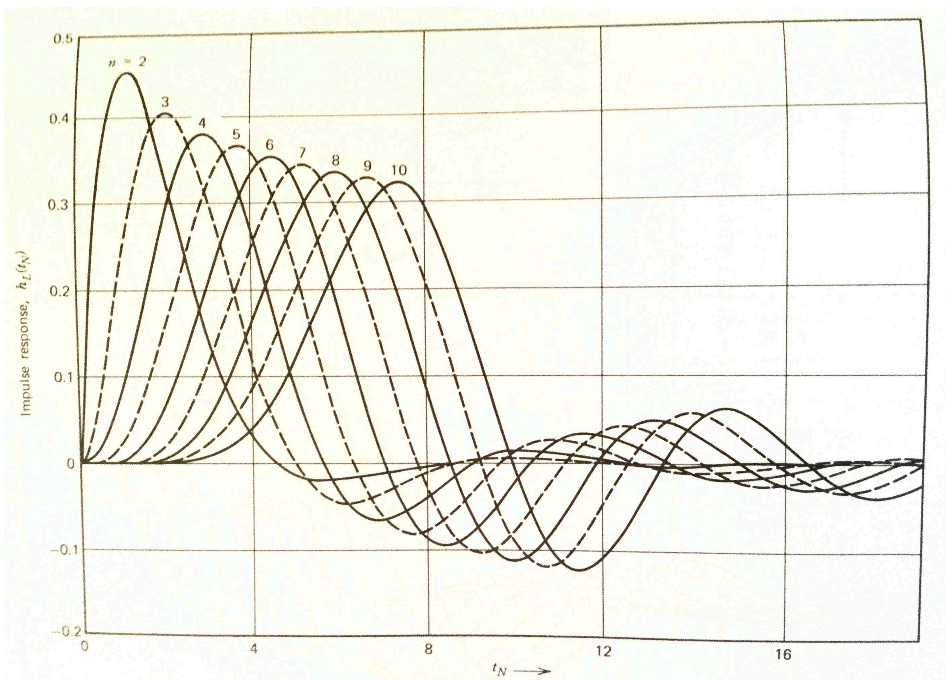
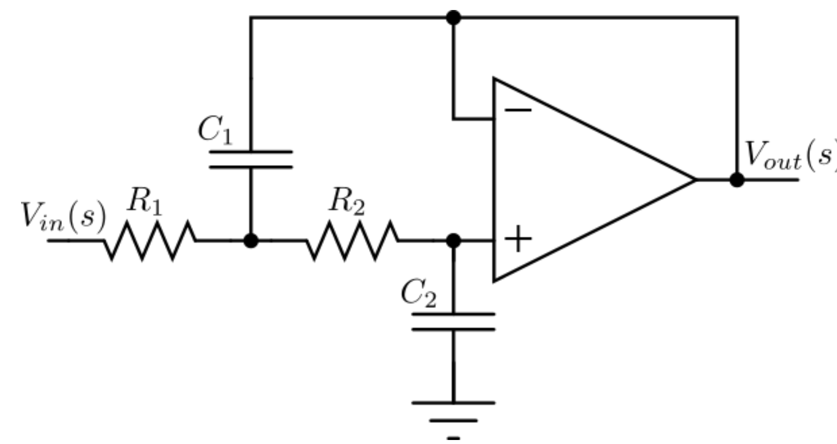


Figure 3-26. Normalized Butterworth LP impulse response.

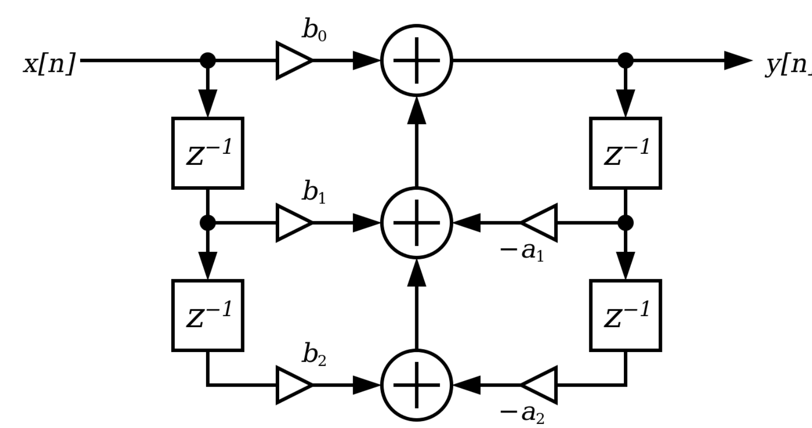
impulse response

Analogue vs. Digital filters

- Analogue filters
  - constructed from analogue circuit components (e.g., resistors, inductors, capacitors)



- Digital filters
  - "hardware" form: set of digital circuits (logic gates, integrated circuits)
  - "software" form: general-purpose micro-computer



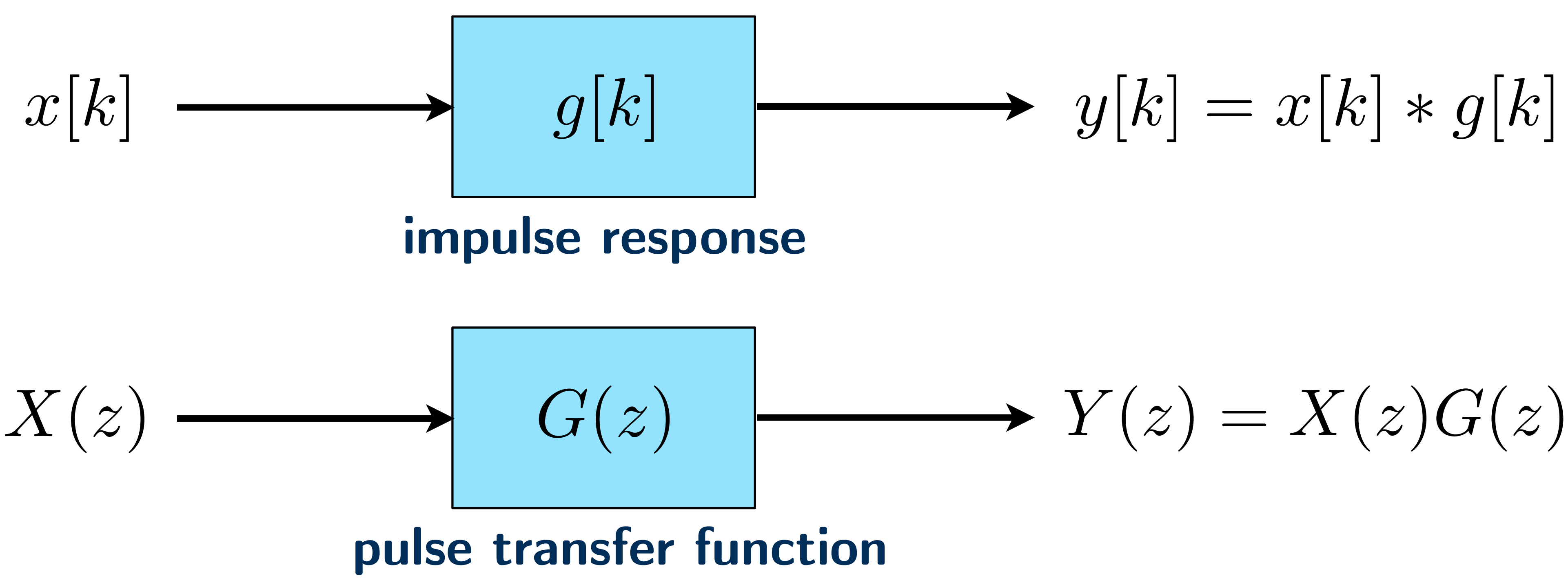

```
>> x = sin([1:100]/10);
>> plot(x)
>> xn = x + randn(1,100)*0.2;
>> plot(xn)
>> y = zeros(1,100);
>> for n=3:100,
y(n) = 0.20657*x(n)+0.41314*x(n-1)+0.207*x(n-2)+0.36953*y(n-1)-0.19582*y(n-2);
end;
>>
>> plot(y)
>> plot(xn)
>> hold on
>> plot(y,'g','linewidth',2)
```

Digital filtering

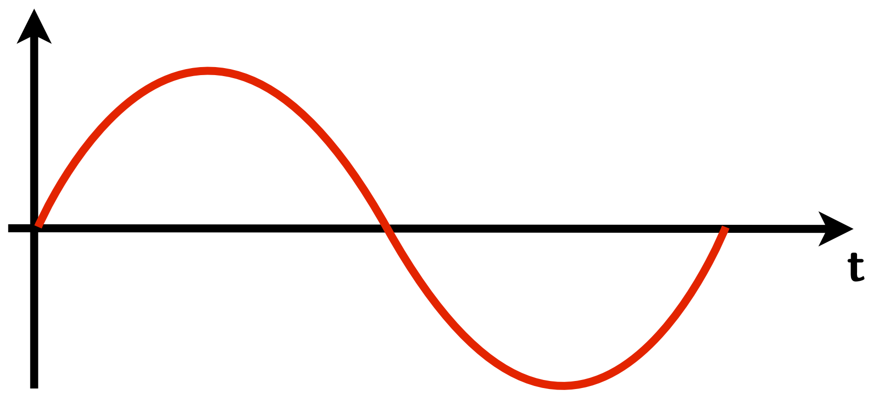
- Can be easily (re-)programmed to implement a number of different filters
- Accuracy only depends on round-off error in the arithmetic
  - hence is predictable and performance known a priori
  - can meet very tight specifications on frequency response
- Widespread use of mini- and micro-computers increased number of digital signals stored and processed
- Robust against noise and change in external environment (e.g., power supply issues, temperature variations)

Digital filtering

- Digital filtering can be done in
  - time domain: convolution with the impulse response
  - frequency domain: multiplication by the desired filter characteristics



The sampling process

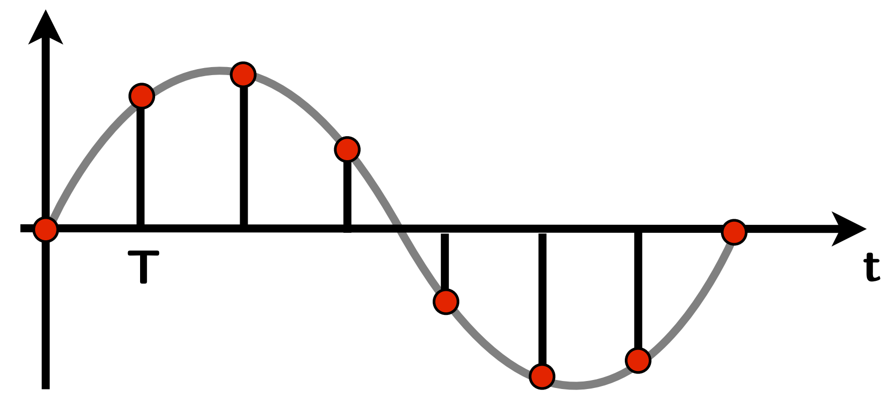


continuous

sampling

f_s — $\frac{\text{sample}}{\text{sec}}$

$T = \frac{1}{f_s}$ — $\frac{\text{sec}}{\text{sample}}$



discrete

$$x_a(t) = A\cos(2\pi f_a t + \phi) = A\cos(\omega_a t + \phi)$$

$$x_d(n) = A\cos(2\pi f_a n T + \phi) = A\cos(2\pi \frac{f_a}{f_s} n + \phi) = A\cos(2\pi f_d n + \phi) = A\cos(w_d n + \phi)$$

$$w_a = 2\pi f_a$$

$\frac{\text{radians}}{\text{sec}}$ $\quad$ $\frac{\text{cycle}}{\text{sec}}$(Hz)

$$f_d = \frac{f_a}{f_s}$$

$$\omega_d = \omega_a T$$

$$w_d = 2\pi f_d$$

$\frac{\text{radians}}{\text{sample}}$ $\quad$ $\frac{\text{cycle}}{\text{sample}}$

$$-\frac{\pi}{T} \leq \omega_a \leq \frac{\pi}{T}$$

$$-\frac{1}{2} f_s = -\frac{1}{2T} \leq f_a \leq \frac{1}{2T} \boxed{= \frac{1}{2} f_s}$$

Nyquist frequency

$$-\pi \leq \omega_d \leq \pi$$

$$-\frac{1}{2} \leq f_d \leq \frac{1}{2}$$

Aliasing

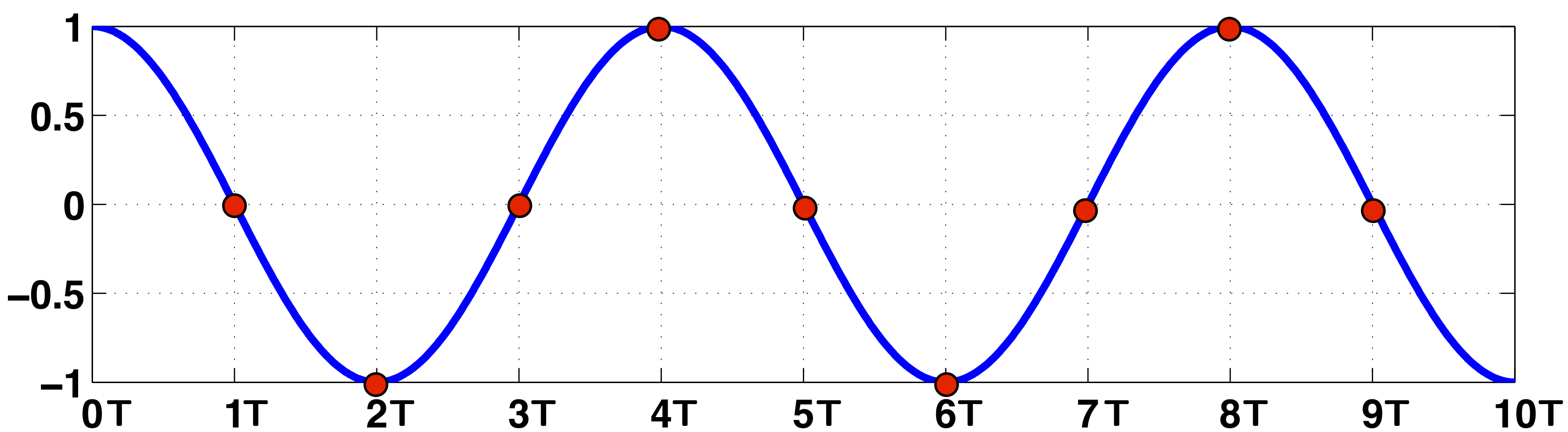
$$f(t) = \cos(\frac{\pi}{2}\frac{t}{T}) \quad \Rightarrow \quad f_d = \frac{1}{4}$$

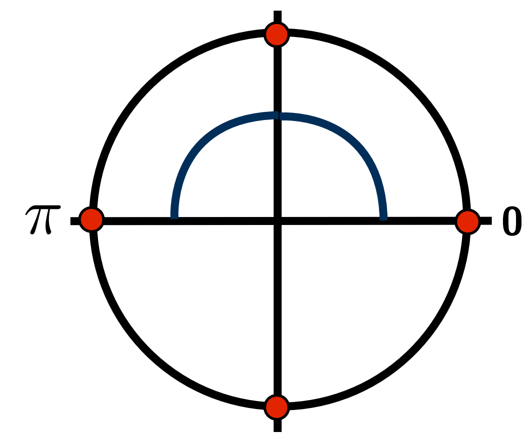
$$f(t) = \cos(\frac{5\pi}{2}\frac{t}{T}) \quad \Rightarrow \quad f_d = \frac{5}{4}$$

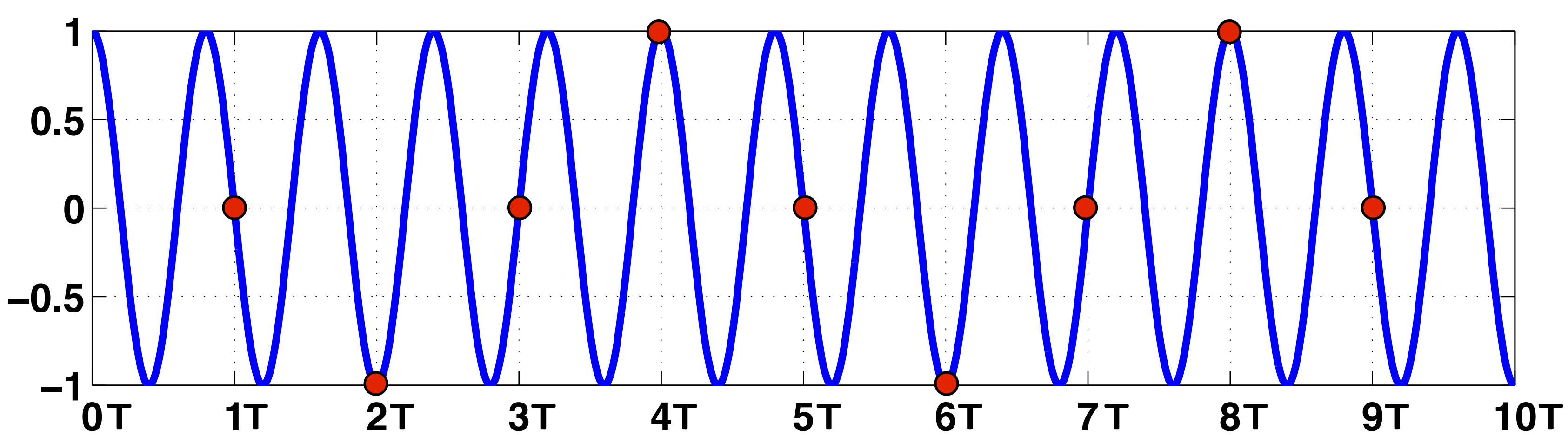
aliasing frequencies:

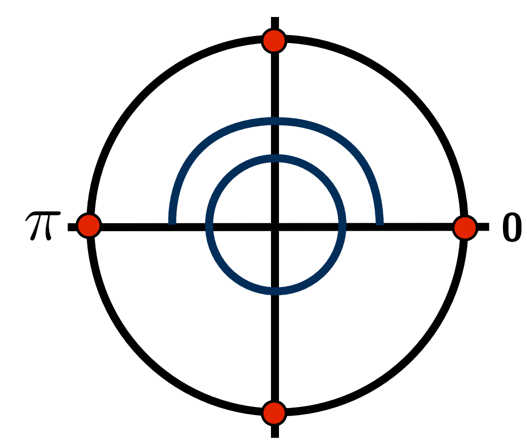
$$f_a = \frac{0.25 \pm k}{T} \text{ Hz}$$

$$(k = 1, 2, \dots)$$









Aliasing

- Sampling in time results in repeated spectrum in frequency

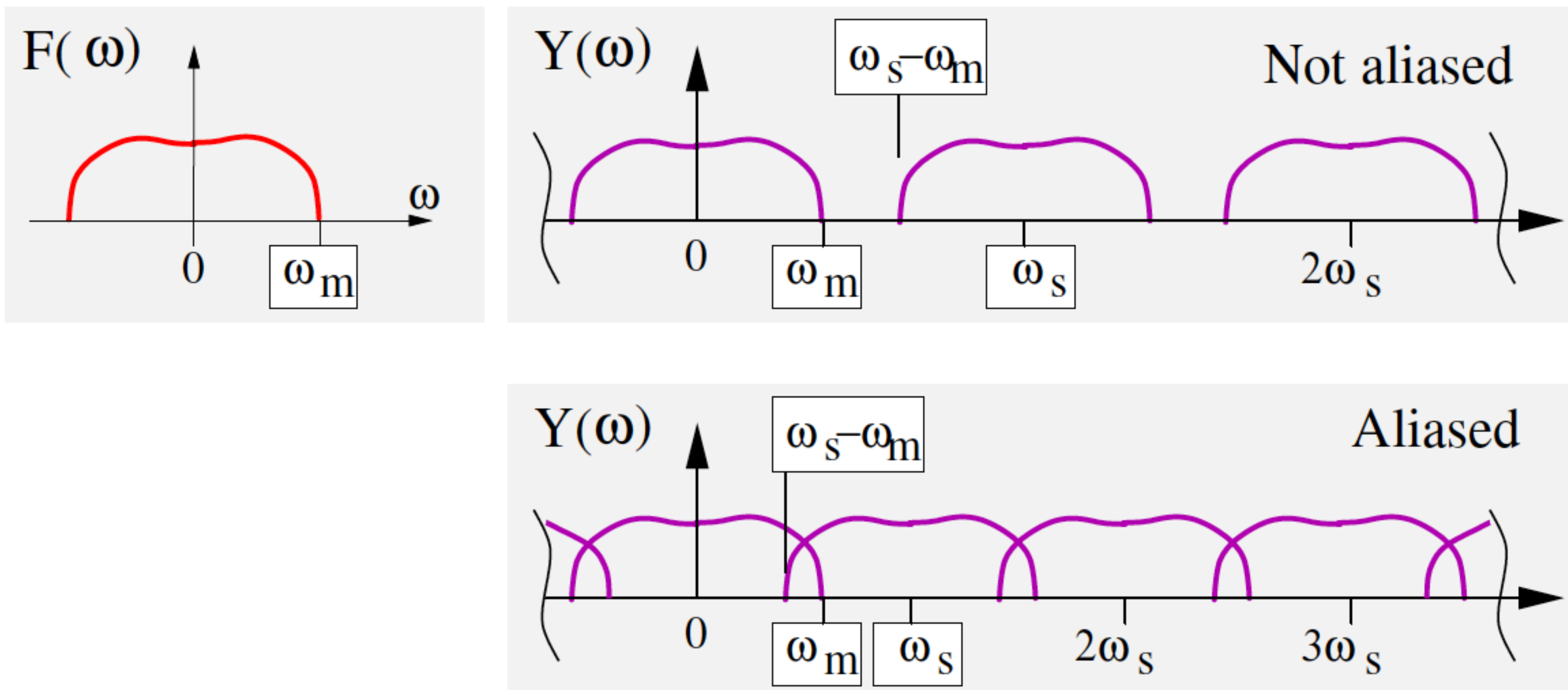
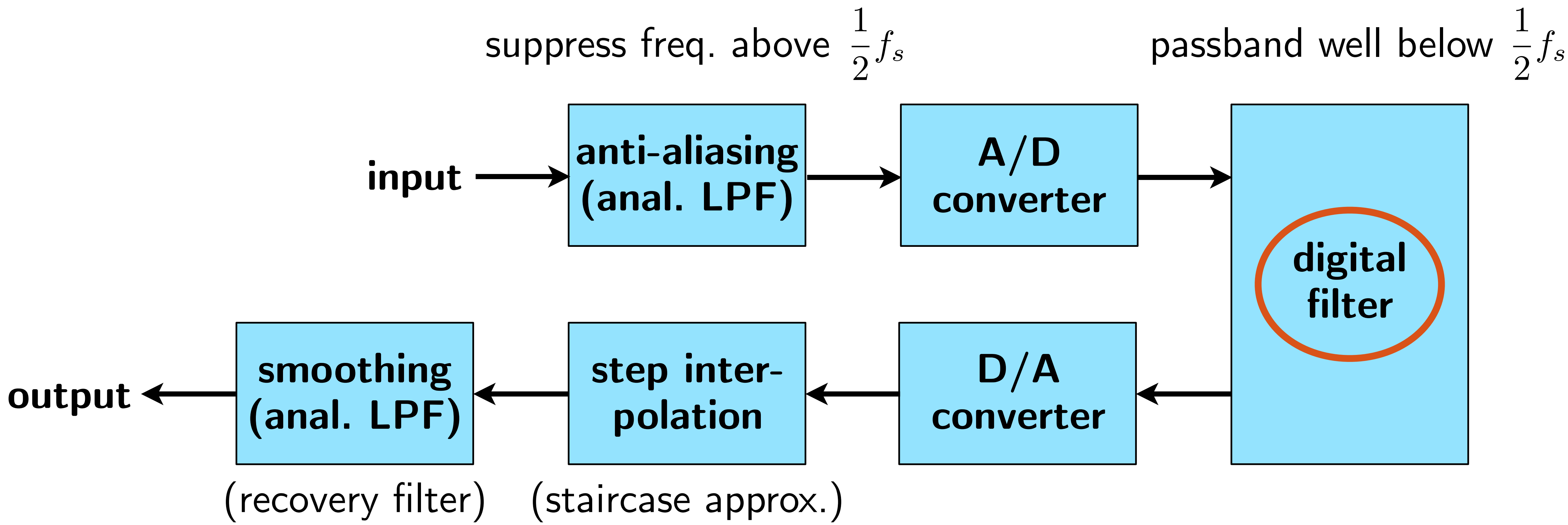
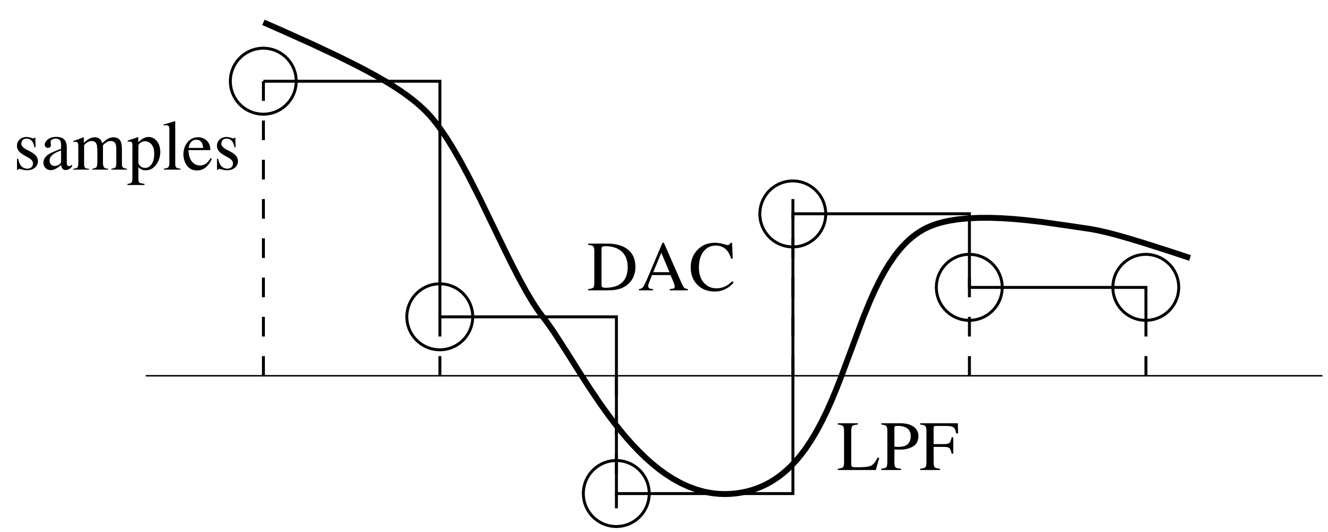


Figure 4.6: Aliasing in the frequency domain

(from lecture notes by David Murray)

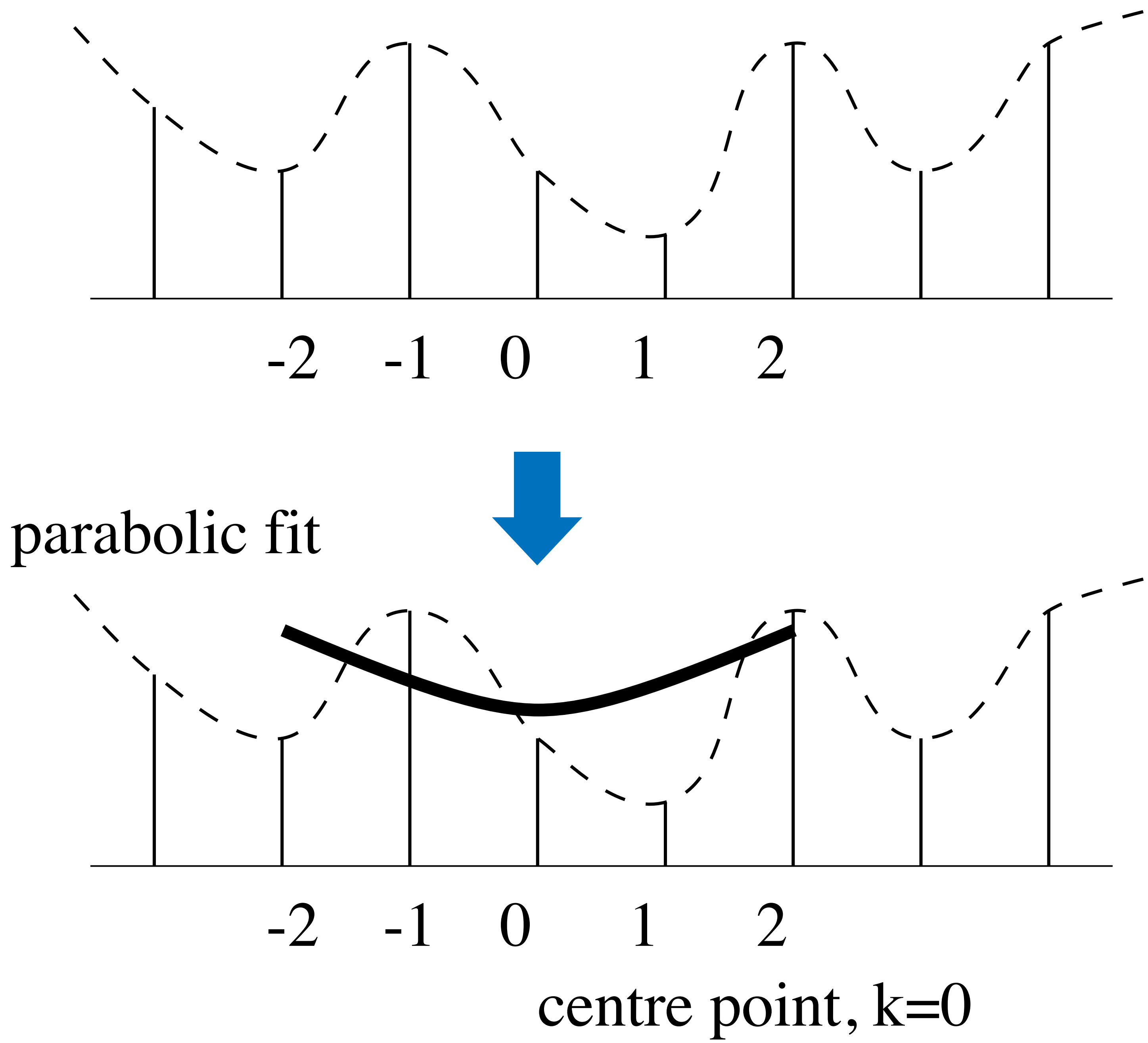
Digital filtering and reconstruction





Digital filtering as regression

- Noise reduction: Polynomial fit using least-squares



Parabolic fit

$$p[k] = s_0 + ks_1 + k^2 s_2 \quad \text{for} \quad k = \{-2, -1, 0, 1, 2\}$$

coefficients of the fit

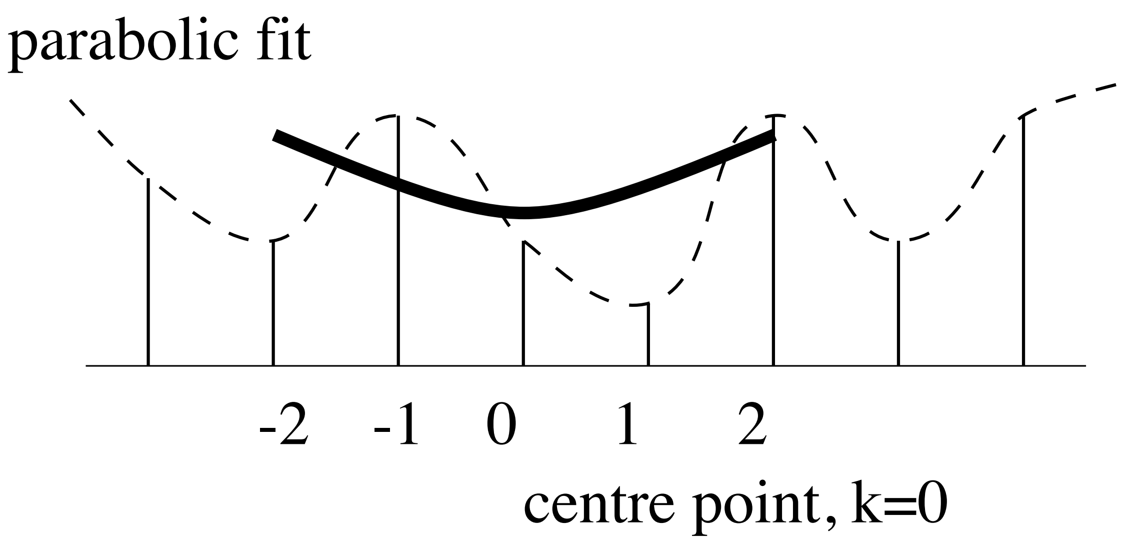
approximation error: $E(s_0, s_1, s_2) = \sum_{k=-2}^{2} (x[k] - [s_0 + ks_1 + k^2 s_2])^2$

$$\frac{\partial E}{\partial s_0} = 0 \Rightarrow 5s_0 + 10s_2 = \sum_{k=-2}^{k=2} x[k] \Rightarrow s_0 = \frac{1}{35}(-3x[-2] + 12x[-1] + 17x[0] + 12x[1] - 3x[2])$$

$$\frac{\partial E}{\partial s_1} = 0 \Rightarrow 10s_1 = \sum_{k=-2}^{k=2} kx[k] \Rightarrow s_1 = \frac{1}{10}(-2x[-2] - x[-1] + x[1] + 2x[2])$$

$$\frac{\partial E}{\partial s_2} = 0 \Rightarrow 10s_0 + 34s_2 = \sum_{k=-2}^{k=2} k^2 x[k] \Rightarrow s_2 = \frac{1}{14}(2x[-2] - x[-1] - 2x[0] - x[1] + 2x[2])$$

Parabolic fit



$$
\begin{aligned}
p[k] \,|_{k=0} &= s_0 + k s_1 + k^2 s_2 \,|_{k=0} = s_0 \\
&= \frac{1}{35}(-3x[-2] + 12x[-1] + 17x[0] + 12x[1] - 3x[2])
\end{aligned}
$$

- the parabola coefficient s_0 is the filtering output
- it provides a smoothed approximation of each set of five data points

Parabolic fit

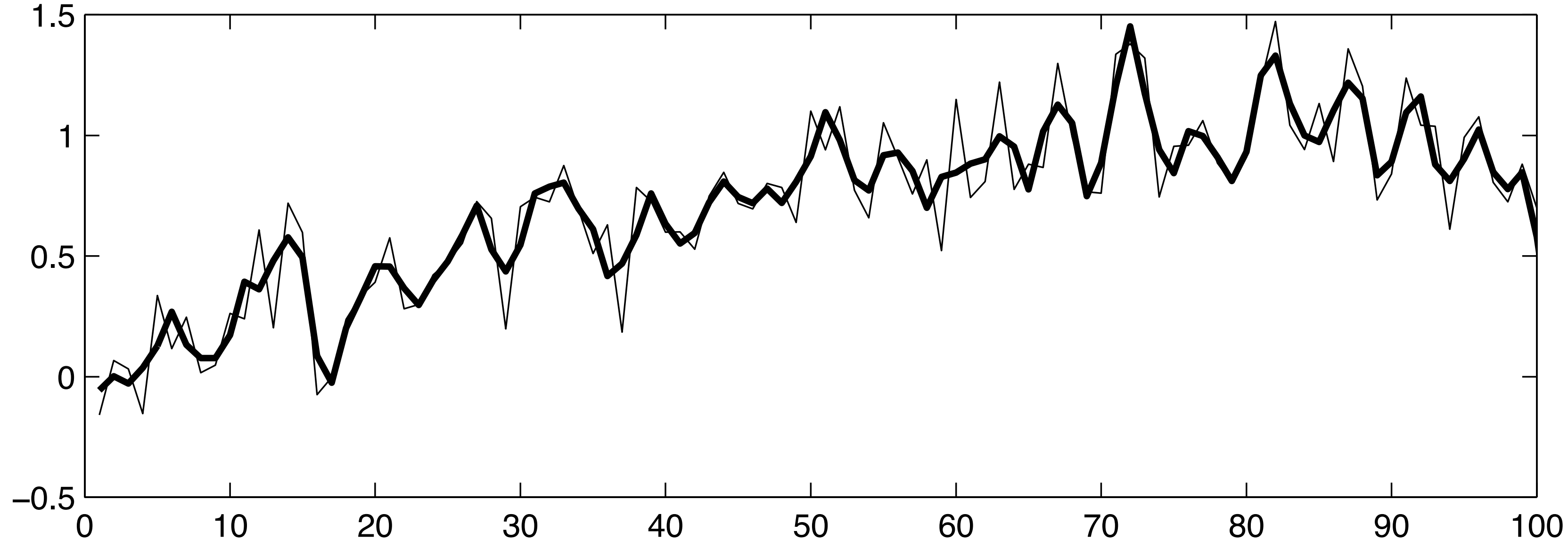


Fig 2.5: *Noisy data (thin line) and 5-point parabolic filtered (thick line).*

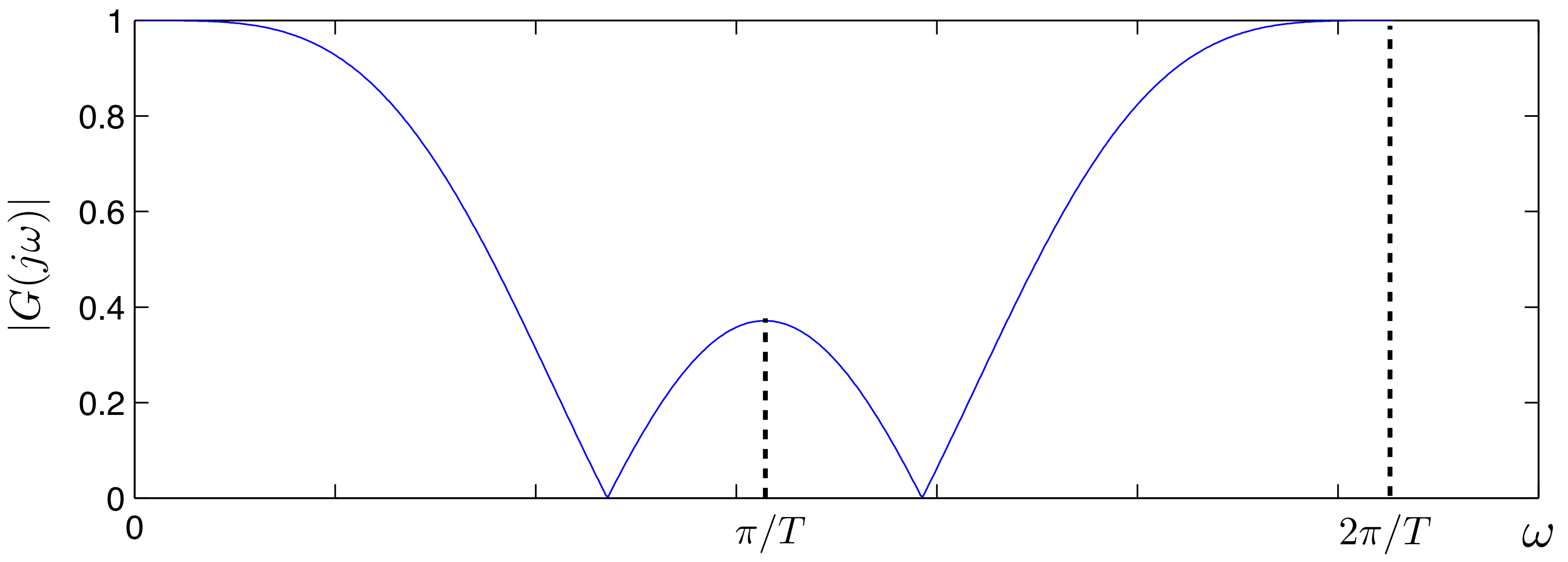


Fig 2.6: *Frequency response of the 5-point parabolic filter.*

Parabolic fit

- The parabolic filter we just considered is
  - a low-pass filter (LPF)
  - a **non-causal** filter:

$$y[k] = \frac{1}{35}(\boxed{-3x[k+2] + 12x[k+1]} + 17x[k] + 12x[k-1] - 3x[k-2])$$

$$\Big\downarrow \text{ delay by 2T}$$

$$y[k] = \frac{1}{35}(-3x[k] + 12x[k-1] + 17x[k-2] + 12x[k-3] - 3x[k-4])$$

  - a **non-recursive** filter: $y[k] = \sum_{i=0}^{N} a_i x[k-i]$

Impulse response of digital filters

- The equation $y[k] = \sum_{i=0}^{N} a_i x[k-i]$ represents a **discrete convolution** of the input data with the filter coefficients
- **Theorem** The coefficients constitute the **impulse response** of the filter.

Proof Let $x[k] = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases}$

Then $y[k] = \sum_{i} a_i x[k-i] = a_k x[0] = a_k$

Finite-Impulse Response (FIR): $y[k] = \sum_{i=0}^{N} a_i x[k-i]$

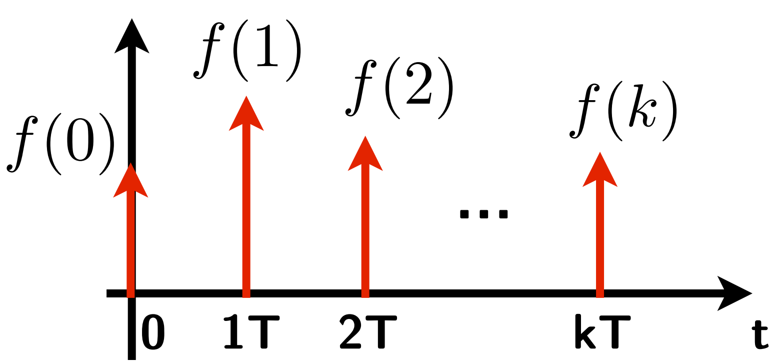
Infinite-Impulse Response (IIR): $y[k] = \sum_{i=0}^{N} a_i x[k-i] + \boxed{\sum_{i=1}^{M} b_i y[k-i]}$

recursive!

The *z*-transform

- The *z*-transform is important in digital filtering
  - it describes frequency-domain properties of discrete (sampled) data
  - it is similar to the Laplace transform in analogue filtering

- Consider the Laplace transform of a discrete function as a succession of impulses



$$F_d(s) = f(0) + f(1)e^{-Ts} + f(2)e^{-2Ts} + \ldots + f(k)e^{-kTs} + \ldots$$

$$\Downarrow \quad z = e^{Ts} = e^{\sigma T} \cdot \boxed{e^{j\omega T}}$$

$$F(z) = f(0) + f(1)z^{-1} + f(2)z^{-2} + \ldots f(k)z^{-k} + \ldots$$

z may be thought of as a **shift operator**

The z-transform

- For many functions, the infinite series can be represented in "closed-form" as the ratio of two polynomials in z^{-1}

step function

$$f[k] = \begin{cases} 0, & \text{if } k < 0 \\ 1, & \text{if } k \geq 0 \end{cases} \quad \Rightarrow \quad \begin{aligned} F(z) &= 1 + z^{-1} + z^{-2} + \ldots + z^{-k} + \ldots \\ &= \frac{1}{1 - z^{-1}} \quad (|z^{-1}| < 1) \end{aligned}$$

decaying exponential

$$f(t) = e^{-\alpha t} \longrightarrow f[k] = e^{-\alpha kT} \quad \Rightarrow \quad \begin{aligned} F(z) &= 1 + e^{-\alpha T} z^{-1} + e^{-\alpha 2T} z^{-2} + \ldots + e^{-\alpha kT} z^{-k} + \ldots \\ &= \frac{1}{1 - e^{-\alpha T} z^{-1}} \end{aligned}$$

sinusoid

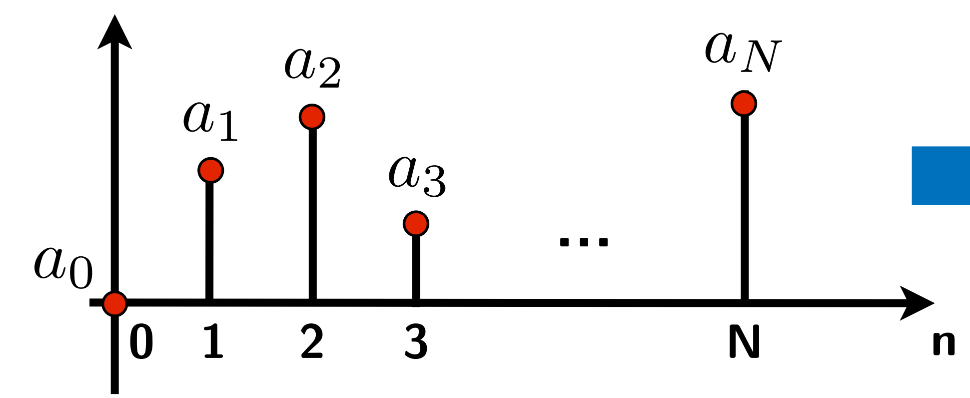
$$f(t) = \cos \omega t \longrightarrow$$

$$f[k] = \cos k\omega T = \frac{e^{jk\omega T} + e^{-jk\omega T}}{2} \quad \Rightarrow \quad \begin{aligned} F(z) &= \frac{1}{2}\left(\frac{1}{1 - e^{j\omega T} z^{-1}} + \frac{1}{1 - e^{-j\omega T} z^{-1}}\right) \\ &= \frac{1 - \cos \omega T z^{-1}}{1 - 2\cos \omega T z^{-1} + z^{-2}} \end{aligned}$$

Pulse transfer function (PTF)

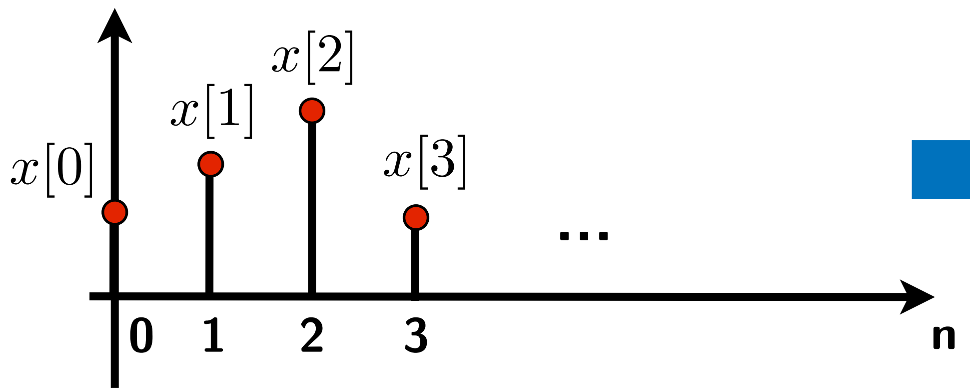
- PTF is ratio of *z*-transform of output to that of input

- Consider an FIR filter with the impulse response



$$G(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_i z^{-i} + \ldots + a_N z^{-N}$$

- Consider an input $x[n]$



$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots + x[k]z^{-k} + \ldots$$

Pulse transfer function (PTF)

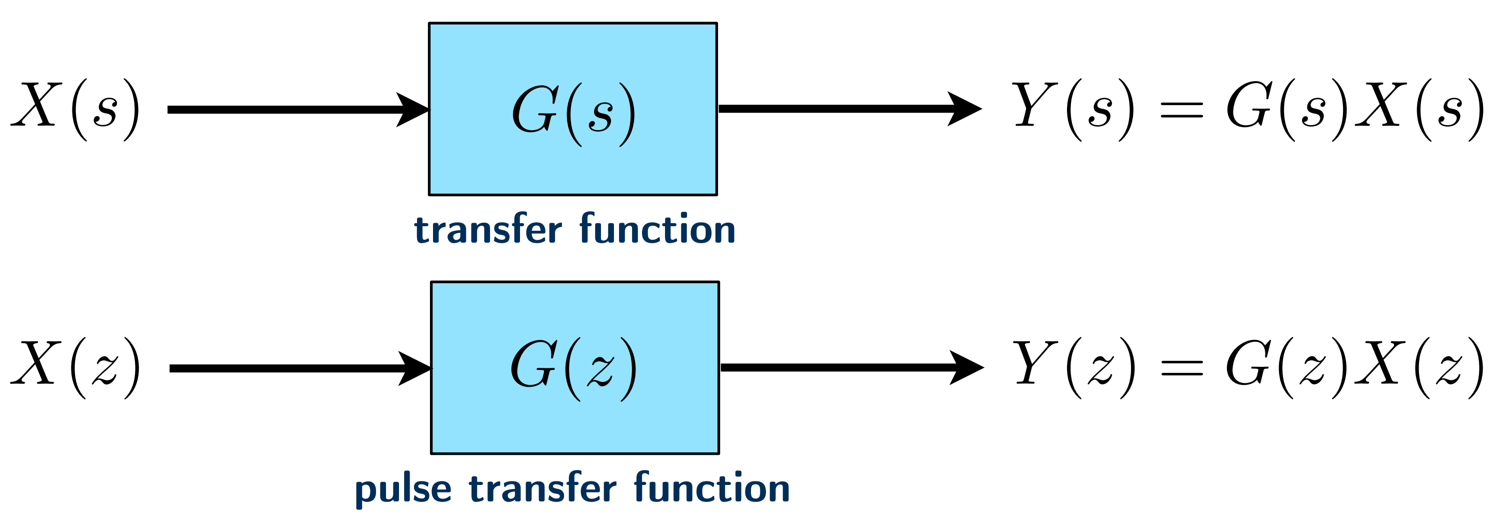
the product $G(z)X(z)$ is

$$G(z)X(z) = (a_0 + a_1 z^{-1} + \ldots + a_i z^{-i} + \ldots + a_N z^{-N})(x[0] + x[1]z^{-1} + \ldots + x[k]z^{-k} + \ldots)$$

in which the coefficient for z^{-k} is

$$a_0 x[k] + a_1 x[k-1] + \ldots + a_i x[k-i] + \ldots + a_N x[k-N]$$

this is again a discrete convolution that gives the **output** $y[k]$, and therefore $Y(z) = G(z)X(z)$: similar to the transfer function!



Pulse transfer function (PTF)

- PTF is the *z*-transform of impulse response
- For non-recursive filters

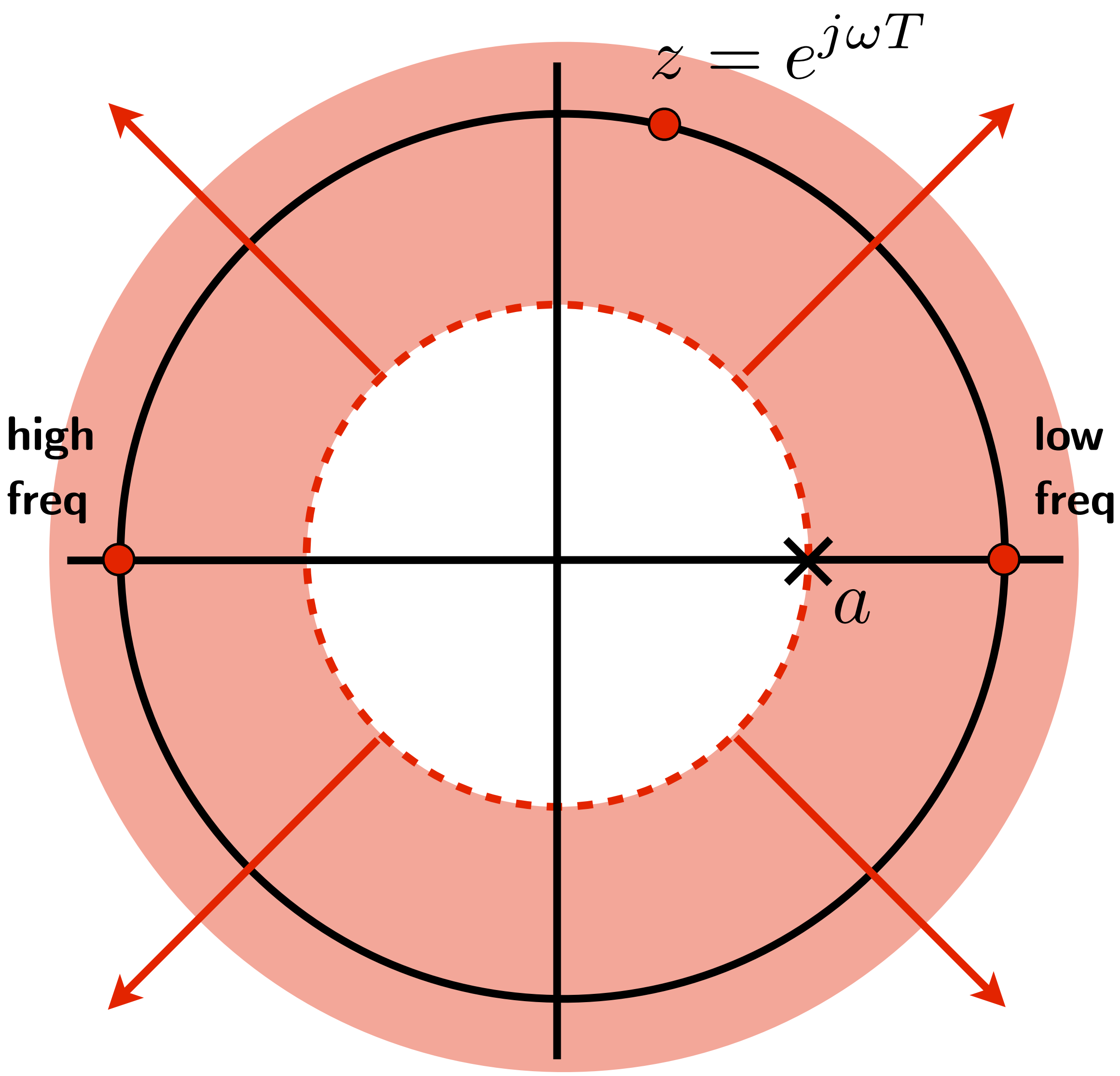
$$G(z) = \sum_{i=0}^{N} a_i z^{-i} \quad \Rightarrow \quad y[k] = \sum_{i=0}^{N} a_i x[k-i]$$

- For recursive filters

$$Y(z) = \sum_{i=0}^{N} a_i z^{-i} X(z) + \sum_{i=1}^{M} b_i z^{-i} Y(z)$$

$$G(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{N} a_i z^{-i}}{1 - \sum_{i=1}^{M} b_i z^{-i}} \quad \Rightarrow \quad y[k] = \sum_{i=0}^{N} a_i x[k-i] + \sum_{i=1}^{M} b_i y[k-i]$$

The z-transform and LTI system



z-plane

$$G(z) = \sum_{-\infty}^{\infty} x[n] z^{-n} < \infty$$

$x[n] = a^n u[n]$ where $0 < a < 1$

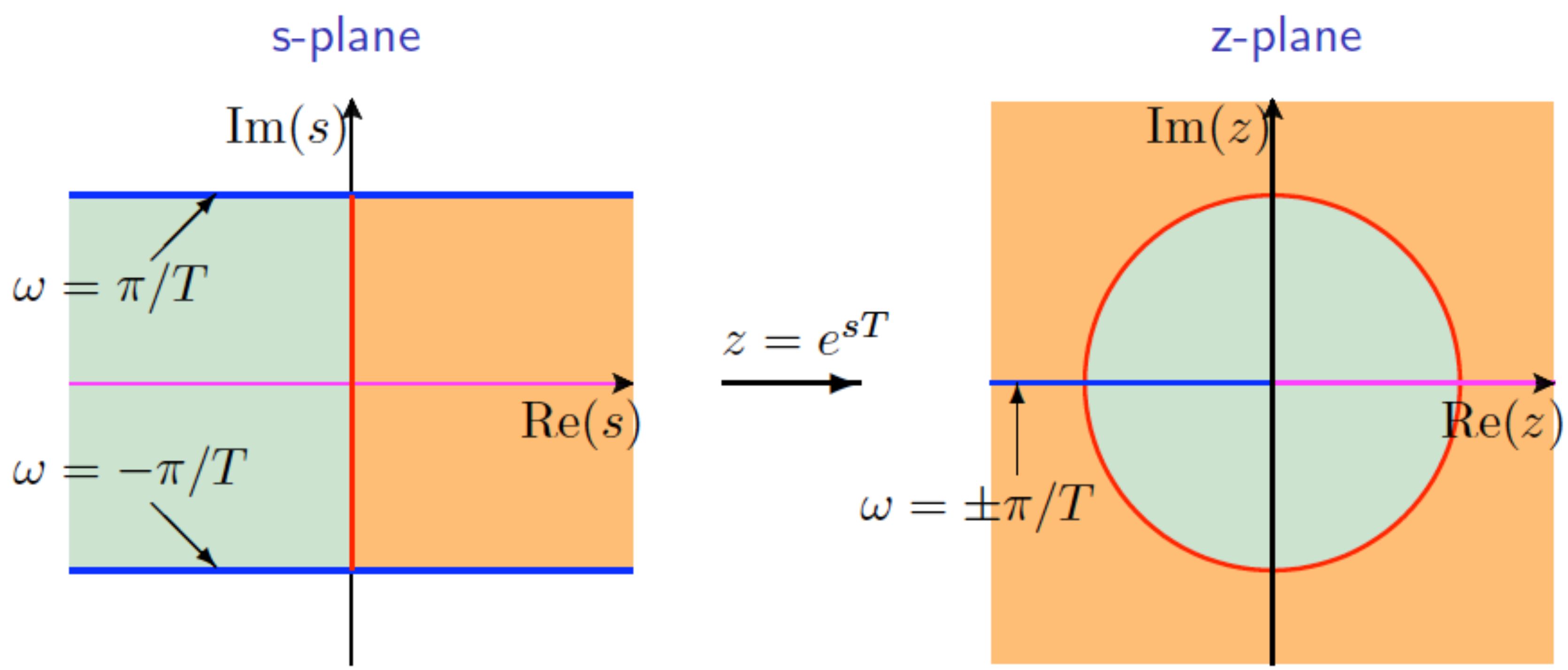
region of convergence (ROC): $|z| > |a|$

$$X(z) = \frac{1}{1 - az^{-1}}$$

- **causal system:** if the ROC extends outward from the outmost pole
- **stable system:** ROC includes the unit circle
- **causal and stable system:** all poles must be inside the unit circle

Mapping from *s*-plane to *z*-plane

$$s = \sigma + j\omega \xrightarrow{z = e^{sT}} z = e^{\sigma T} e^{j\omega T}$$



(from lecture slides by Mark Cannon)

imaginary axis ($\sigma = 0$)	unit circle ($\lvert z \rvert = 1$)
left-half plane ($\sigma < 0$)	inside unit circle ($\lvert z \rvert < 1$)
right-half plane ($\sigma > 0$)	outside unit circle ($\lvert z \rvert > 1$)
poles **in left-half plane** for stability	poles **inside unit circle** for stability

Example

- What's the condition for the following filter to be stable?

$$y[k] = x[k-1] + \alpha y[k-1]$$

$$\Downarrow$$

$$Y(z)(1 - \alpha z^{-1}) = z^{-1}X(z)$$

$$\Downarrow$$

$$G(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - \alpha z^{-1}} = \frac{1}{z - \alpha}$$

hence for the filter to be stable we need $|\alpha| < 1$.

Frequency response of a digital filter

- **Theorem**

  The frequency response of a digital filter can be obtained by evaluating the PTF on the unit circle ($z = e^{j\omega T}$)

- **Proof** Consider the general form of a digital filter

$$y[k] = \sum_{i=0}^{\infty} a_i x[k-i]$$

Consider an input $\cos(\omega t + \theta)$ sampled at $t = 0, T, \ldots, kT$

therefore

$$x[k] = \cos(\omega kT + \theta) = \frac{1}{2}\{e^{j(\omega kT+\theta)} + e^{-j(\omega kT+\theta)}\}$$

Frequency response of a digital filter

$$y[k] = \frac{1}{2}\sum_{i=0}^{\infty} a_i e^{j\{\omega[k-i]T+\theta\}} + \frac{1}{2}\sum_{i=0}^{\infty} a_i e^{-j\{\omega[k-i]T+\theta\}}$$

$$= \frac{1}{2}e^{j(\omega kT+\theta)}\boxed{\sum_{i=0}^{\infty} a_i e^{-j\omega iT}} + \frac{1}{2}e^{-j(\omega kT+\theta)}\boxed{\sum_{i=0}^{\infty} a_i e^{j\omega iT}}$$

N.B. $\displaystyle\sum_{i=0}^{\infty} a_i e^{-j\omega iT} = \sum_{i=0}^{\infty} a_i (e^{j\omega T})^{-i} = \sum_{i=0}^{\infty} a_i z^{-i}|_{z=e^{j\omega T}} = G(z)|_{z=e^{j\omega T}}$

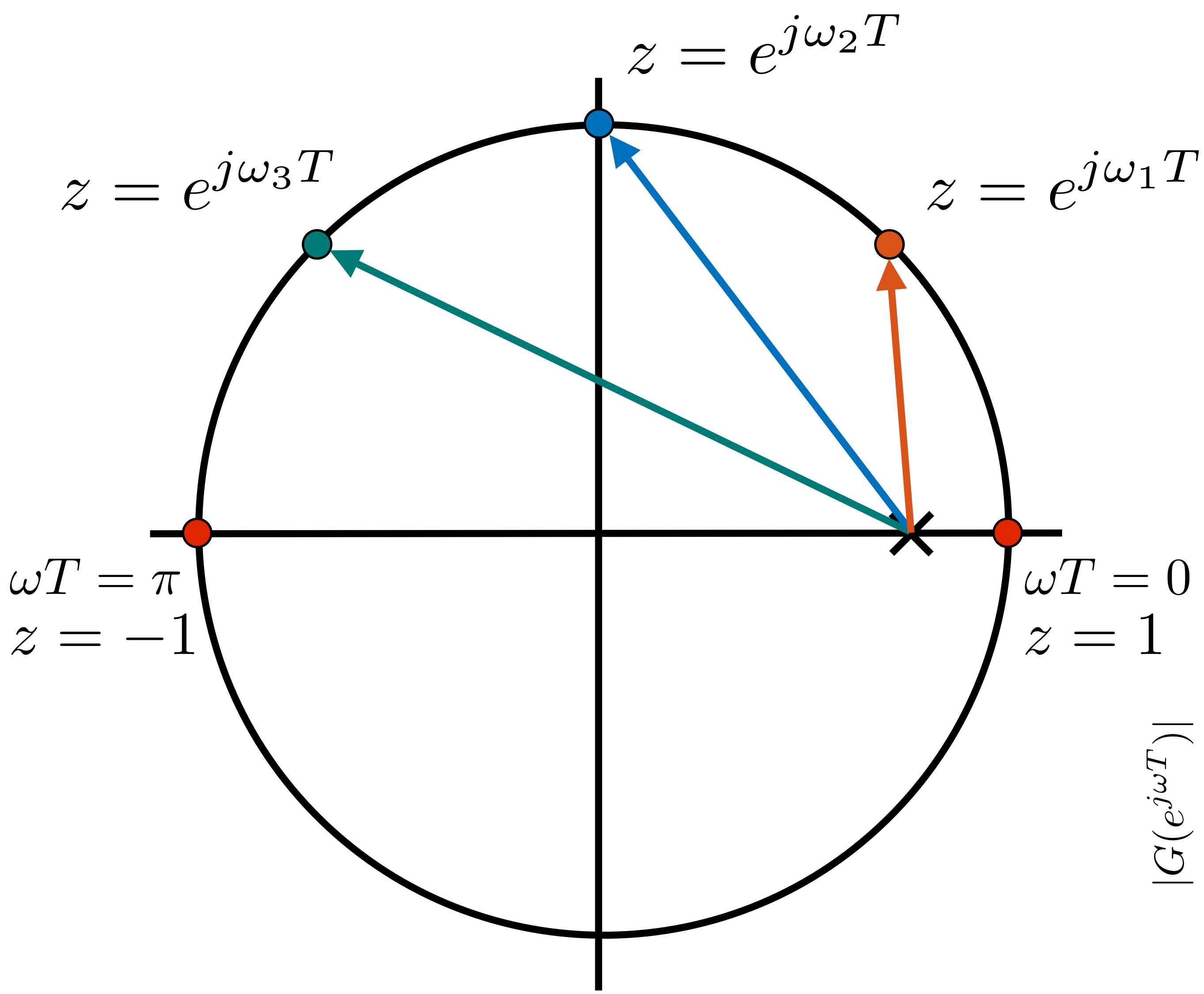
let $\quad G(z)|_{z=e^{j\omega T}} = \boxed{Ae^{j\phi}} \quad$ then $\quad \displaystyle\sum_{i=0}^{\infty} a_i e^{j\omega iT} = \boxed{Ae^{-j\phi}}$

hence $\quad \displaystyle y[k] = \frac{1}{2}e^{j(\omega kT+\theta)}Ae^{j\phi} + \frac{1}{2}e^{-j(\omega kT+\theta)}Ae^{-j\phi}$

or $\quad y[k] = A\cos(\omega kT + \theta + \phi) \quad$ while $\quad x[k] = \cos(\omega kT + \theta)$

thus A and ϕ represent the gain and phase of the frequency response, i.e., the frequency response is $G(z)|_{z=e^{j\omega T}}$.

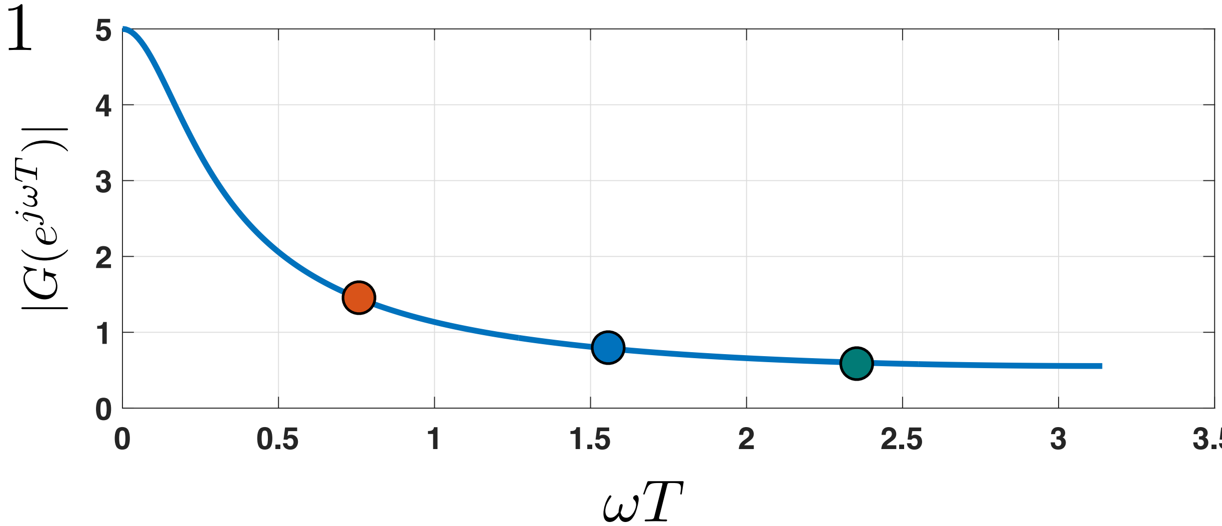
Example



z-plane

$$G(z) = \frac{1}{z - 0.8}$$

$$G(z)|_{z=e^{j\omega T}} = \frac{1}{e^{j\omega T} - 0.8}$$



Example

- Consider the 5-point parabolic filter

$$
\begin{aligned}
y[k] &= \frac{1}{35}(-3x[k] + 12x[k-1] + 17x[k-2] + 12x[k-3] - 3x[k-4]) \\
Y(z) &= \frac{1}{35}(-3 + 12z^{-1} + 17z^{-2} + 12z^{-3} - 3z^{-4})X(z) \\
G(z)|_{z=e^{j\omega T}} &= \frac{1}{35}(-3 + 12e^{-j\omega T} + 17e^{-2j\omega T} + 12e^{-3j\omega T} - 3e^{-4j\omega T}) \\
&= \frac{1}{35}e^{-2j\omega T}(17 + 24\cos\omega T - 6\cos 2\omega T)
\end{aligned}
$$

Example

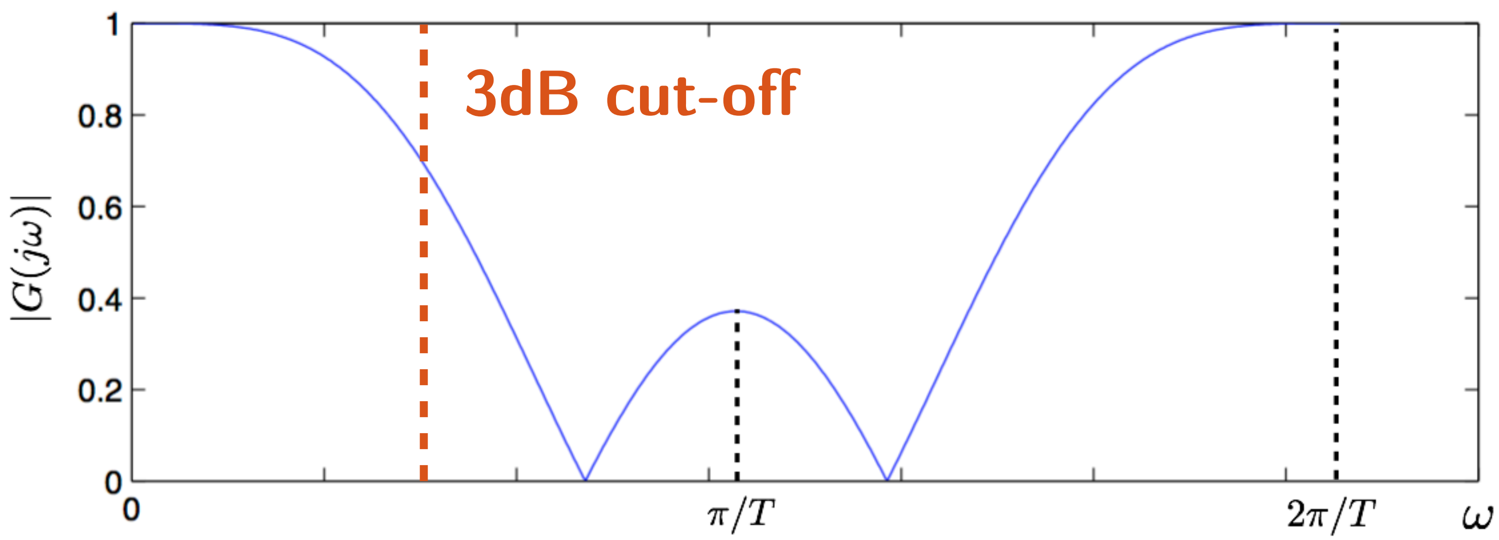
$$G(z)|_{z=e^{j\omega T}} = \boxed{\frac{1}{35}\boxed{e^{-2j\omega T}}(17 + 24\cos\omega T - 6\cos 2\omega T)}$$

therefore $$|G(e^{j\omega T})| = \frac{1}{35}|17 + 24\cos\omega T - 6\cos 2\omega T|$$

$$\omega T = 0 \rightarrow |G(e^{j\omega T})| = 1$$

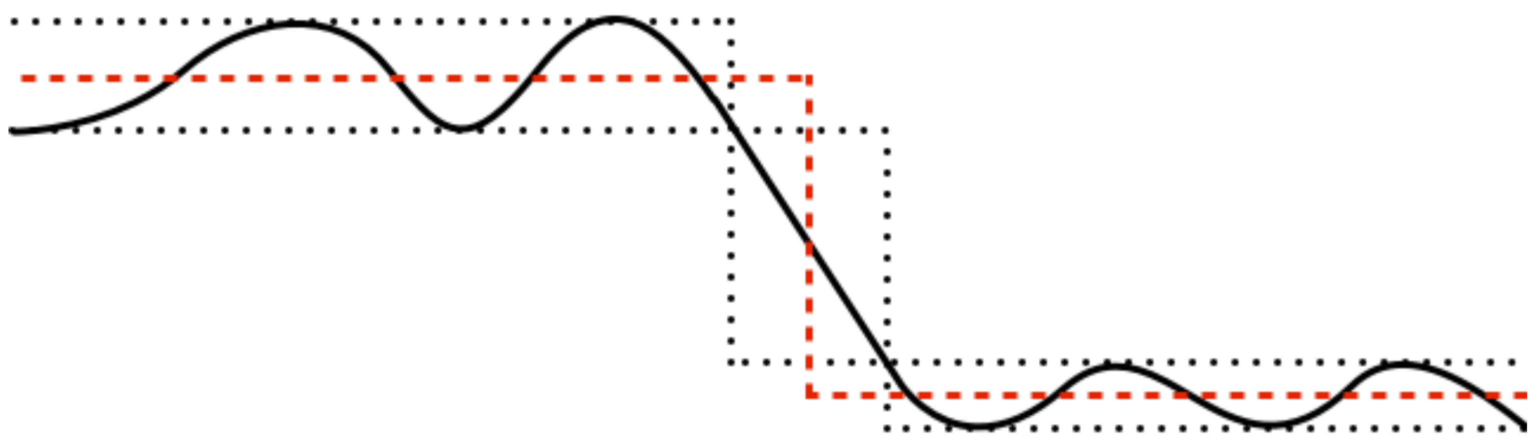
$$\omega T \approx 0.48\pi \text{ (i.e., } f/f_s = 0.24) \rightarrow |G(e^{j\omega T})| = 0.707$$

$\angle G(e^{j\omega T}) = -2\omega T$ (**linear-phase** - all frequencies delayed by 2T)



Design of digital filters

- Three basic steps
  - specification of desired frequency response
  - **approximation** of the specification using a **causal** discrete-time system
  - realisation of the system using finite-precision arithmetic



- Different design techniques for FIR and IIR filters

Continuous vs. Discrete system

continuous	**discrete**
linear differential equation (impulse response)	linear difference equation (impulse response)
convolution integral	convolution sum
Laplace transform (transfer function)	z-transform (pulse transfer function)
Frequency response (imaginary axis -> Fourier transform)	Frequency response (unit circle -> discrete-time Fourier transform)
analogue filter	digital filter

Fourier transform for discrete-time signals

- We have introduced **Fourier series (FS)** for continuous **periodic signals**

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j2\pi n f_0 t}dt \qquad x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_0 t}$$

and **Fourier transform (FT)** for continuous **aperiodic signals**

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)e^{j\omega t}d\omega$$

- In digital signal processing and filtering we need to deal with discrete-time signals - what about the discrete-time counterparts of FS and FT?

Discrete Fourier transform

DFT:
$$\boxed{F[n] = \sum_{k=0}^{N-1} f[k] e^{-j2\pi \frac{n}{N} k}} \quad \text{for} \quad n = 0, 1, ..., N-1$$

FS:
$$\boxed{C_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t) e^{-j2\pi \frac{n}{T_0} t} dt} \quad \text{for} \quad n \in (-\infty, ..., \infty)$$

$$\begin{pmatrix} F[0] \\ F[1] \\ F[2] \\ \vdots \\ F[N-1] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & W & W^2 & W^3 & \dots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \dots & W^{N-2} \\ 1 & W^3 & W^6 & W^9 & \dots & W^{N-3} \\ \vdots & & & & & \\ 1 & W^{N-1} & W^{N-2} & W^{N-3} & \dots & W \end{pmatrix} \begin{pmatrix} f[0] \\ f[1] \\ f[2] \\ \vdots \\ f[N-1] \end{pmatrix}$$

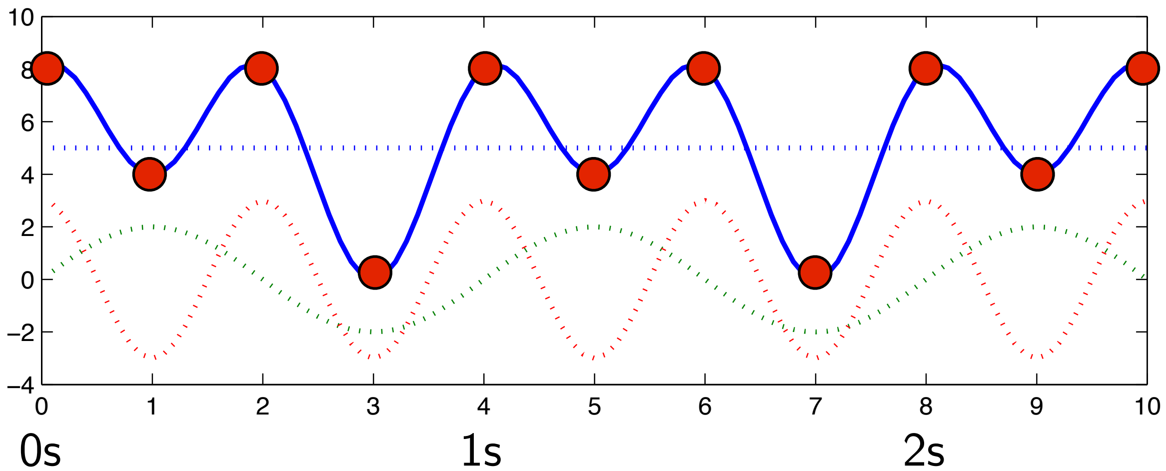
where $W = e^{-j\frac{2\pi}{N}}$ and $W^N = W^{2N} = ... = 1$

IDFT:
$$\boxed{f[k] = \frac{1}{N} \sum_{n=0}^{N-1} F[n] e^{+j\frac{2\pi}{N} nk}}$$

Example

- Consider the following signal

$$f(t) = \underbrace{5}_{\text{dc}} + \underbrace{2\cos(2\pi t - 90^o)}_{1\text{Hz}} + \underbrace{3\cos 4\pi t}_{2\text{Hz}}$$



sample at $f_s = 4$ Hz

$t = kT = k/4$ sec

$$f[k] = 5 + 2\cos(\frac{\pi}{2}k - 90^o) + 3\cos\pi k$$

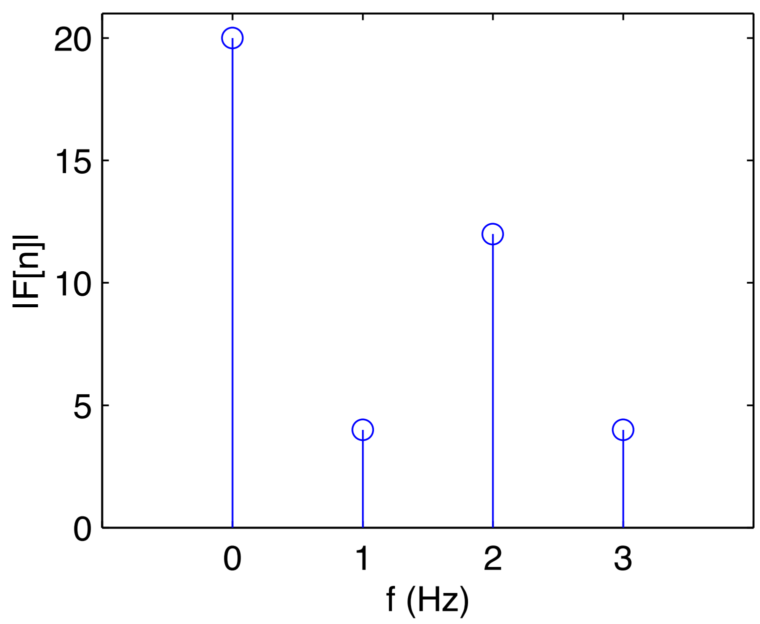
$$f[0] = 8, f[1] = 4, f[2] = 8, f[3] = 0 \quad (N = 4)$$

Example

- Compute DFT

$$F[n] = \sum_{0}^{3} f[k] e^{-j\frac{\pi}{2}nk} = \sum_{k=0}^{3} f[k](-j)^{nk}$$

$$\begin{pmatrix} F[0] \\ F[1] \\ F[2] \\ F[3] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \begin{pmatrix} f[0] \\ f[1] \\ f[2] \\ f[3] \end{pmatrix} = \begin{pmatrix} 20 \\ -j4 \\ 12 \\ j4 \end{pmatrix}$$



The Fourier transform - Four different forms

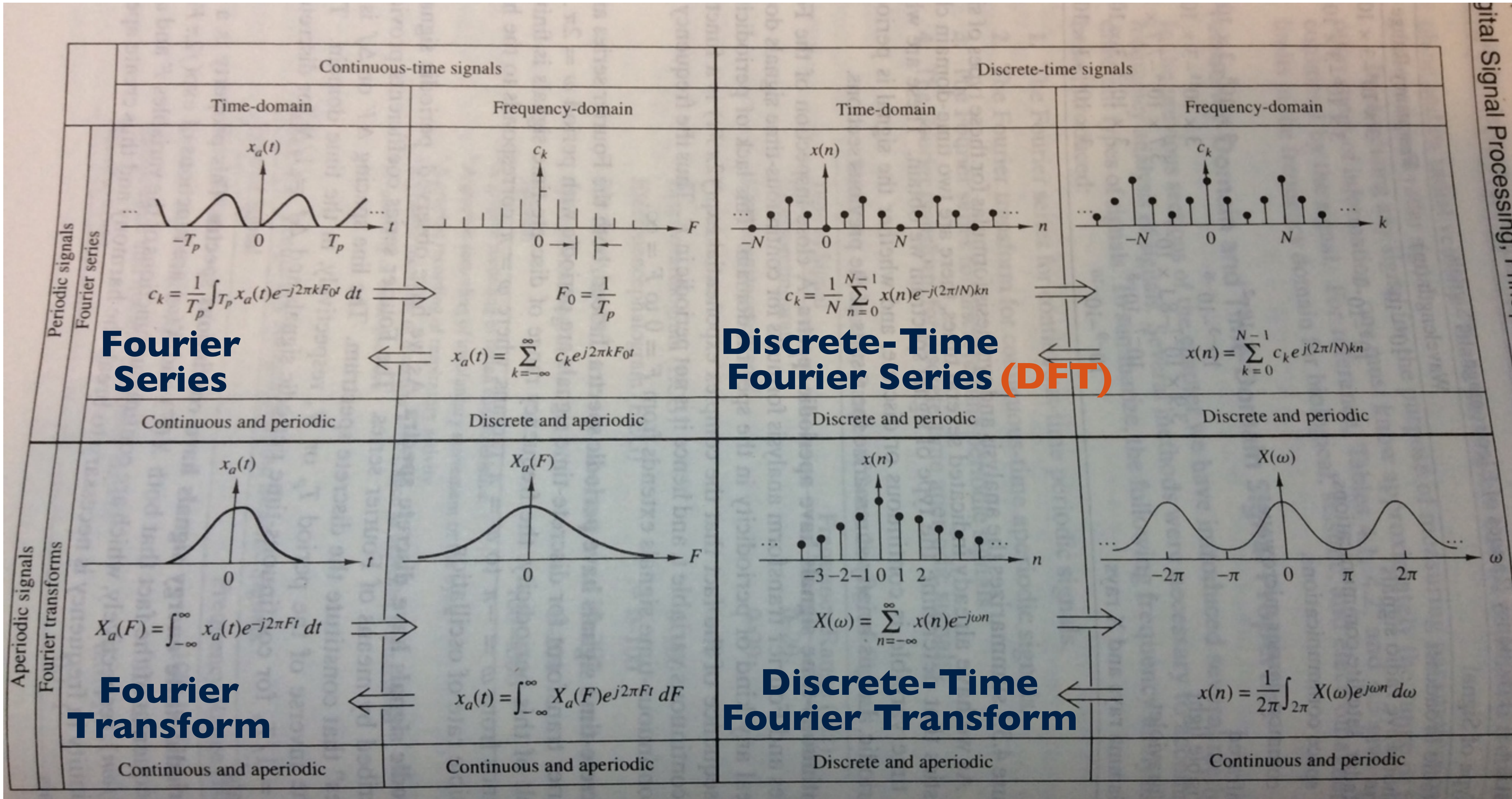


Figure 4.3.1 Summary of analysis and synthesis formulas.

Summary

- LTI systems are of central importance to modern signal processing
- Time- and frequency-domain representations of the system are equivalent; such equivalence is established by the convolution theorem
- Frequency-selective filters are one of the most important signal processing tools
- The DFT, which represents a finite sequence with finite number of coefficients, plays a central role in digital signal processing and filtering