Derivation of the Fourier Transform of the Mask for Optical Heterodyning
Supplement to SIGGRAPH 2007 paper [1]

1 Derivation

This analysis is done for a 1D mask placed in front of a 1D sensor to capture a 2D light field.

Let v be the total distance between the aperture and the sensor and d be the distance between the mask and the sensor. Define $\beta = \frac{d}{v}$. From Figure 1, if we place the 1D code $c(y)$ at a distance d from the sensor, the resulting 2D light field gets attenuated by the 2D mask $m(x, \theta)$ given by

$$m(x,\theta) = c(\beta\theta + (1-\beta)x). \tag{1}$$

As we will derive below, the Fourier transform of the mask lies on a line in the 2D Fourier light field space.

Let $C(f_y)$ be the 1D Fourier transform of $c(y)$

$$C(f_y) = \int_{-\infty}^{\infty} c(y)\exp(-j2\pi f_y y)dy \tag{2}$$

and let $M(f_x, f_\theta)$ be the 2D Fourier transform of $m(x,\theta)$:

$$M(f_x, f_\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(x,\theta)\exp(-j2\pi f_x x)\exp(-j2\pi f_\theta \theta)dxd\theta. \tag{3}$$

We wish to find the expression of $M(f_x, f_\theta)$ in terms of $C(y)$.

Let

$$y = \beta\theta + (1-\beta)x \tag{4}$$

Use auxiliary variable

$$z = (1-\beta)\theta - \beta x \tag{5}$$

Define

$$\mu = \frac{1}{\sqrt{\beta^2 + (1-\beta)^2}} \tag{6}$$

Then

$$\theta = \mu^2(\beta y + (1-\beta)z) \tag{7}$$

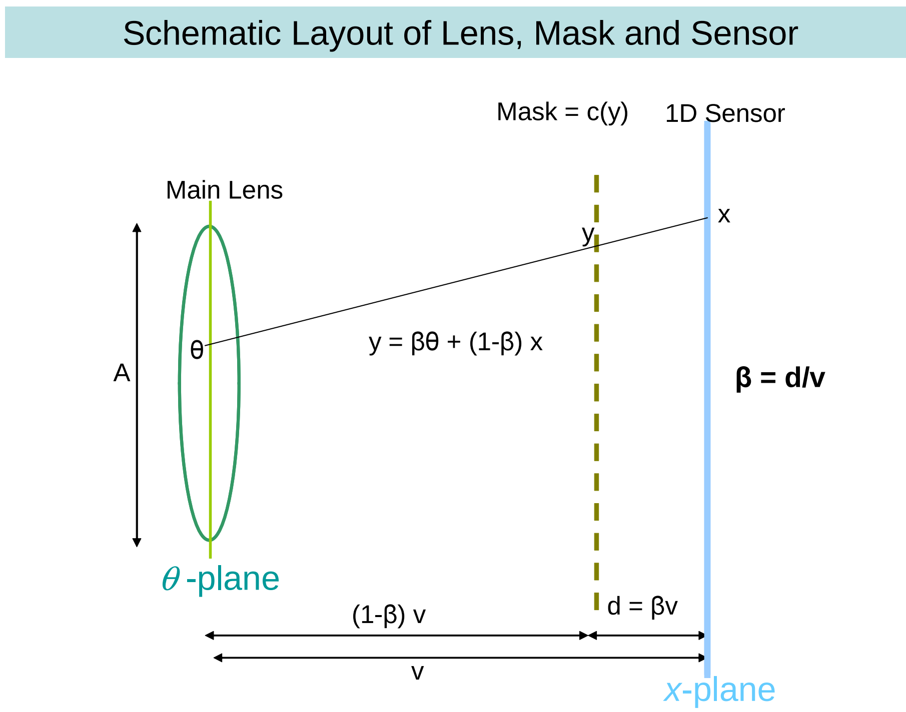

Figure 1: Schematic showing a 1D code place in front of a 1D sensor to capture a 2D light field. The light field is parameterized as twin-plane, with the x plane aligned with the sensor and the θ plane aligned with the aperture.

$$x = \mu^2((1-\beta)y - \beta z) \tag{8}$$

Jacobian

$$J = \det \begin{bmatrix} \frac{\partial\theta}{\partial y} & \frac{\partial\theta}{\partial z} \\ \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} \end{bmatrix} \tag{9}$$

$$J = \mu^2 \tag{10}$$

By change of variables, we have

$$M(f_x, f_\theta) = J \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(y) \exp(-j2\pi f_y y) \exp(-j2\pi f_z z) dy dz \tag{11}$$

Substituting x and θ from (7) and (8) in (3), and comparing common terms, we get

$$f_y = \mu^2 (f_x(1-\beta) + f_\theta \beta) \tag{12}$$

$$f_z = \mu^2(-f_x\beta + f_\theta(1-\beta)) \tag{13}$$

Integrating out z term will give a δ term. Thus

$$M(f_x, f_\theta) = JC(f_y)\delta(f_z) \tag{14}$$

Now substitute $\tan\alpha = \frac{\beta}{1-\beta}$
Then $\sin\alpha = \mu\beta$ and $\cos\alpha = \mu(1-\beta)$
Substituting in equation for f_y, we get

$$f_y = \mu(f_x \cos\alpha + f_\theta \sin\alpha) \tag{15}$$

Simplifying $\delta(f_z)$, we get

$$\delta(f_z) = \delta(f_\theta \cos\alpha - f_x \sin\alpha) \tag{16}$$

Finally

$$M(f_x, f_\theta) = \mu^2 C(\mu(f_x \cos\alpha + f_\theta \sin\alpha))\delta(f_\theta \cos\alpha - f_x \sin\alpha) \tag{17}$$

The δ function constraints the 2D Fourier transform of mask to lie along a line given by $f_\theta \cos\alpha - f_x \sin\alpha = 0$

Using this constraint the above equation can be simplified to

$$M(f_x, f_\theta) = \mu^2 C(\mu\sqrt{f_x^2 + f_\theta^2})\delta(f_\theta \cos\alpha - f_x \sin\alpha). \tag{18}$$

Thus,

$$\tan\alpha = \frac{\beta}{1-\beta} = \frac{d}{v-d}. \tag{19}$$

and

$$\mu = \frac{1}{\sqrt{\beta^2 + (1-\beta)^2}} = \frac{1}{\sqrt{(d/v)^2 + (1-(d/v))^2}} \tag{20}$$

1.1 Practical Design

In a practical design, first α is calculated using the frequency resolution in θ, $f_{\theta R}$ and the bandlimit f_{x0} of the light field in the spatial dimension. $f_{\theta R}$ is relate to the size of the aperture A. $f_{\theta R} = 1/A$.

$$\tan\alpha = \frac{f_{\theta R}}{2f_{x0}} \tag{21}$$

Once we know α and the total distance between the sensor and the aperture v, we can find d using (19). The fundamental frequency can be obtained using (18) by substituting $f_x = 2f_{x0}$ and $f_\theta = f_{\theta R}$

$$f_0 = \mu\sqrt{4f_{x0}^2 + f_{\theta R}^2} \tag{22}$$

In practice, μ is close to 1 and α is $\approx 4-5$ degrees.

References

[1] A. Veeraraghavan, R. Raskar, A. Agrawal, A. Mohan, and J. Tumblin. Dappled photography: Mask enhanced cameras for heterodyned light fields and coded aperture refocusing. *ACM Trans. Graph.*, 26(3), July 2007.