LEARNING THROUGH NEGOTIATING CONCEPTUALLY GENERATIVE PERSPECTIVAL COMPLEMENTARITIES: THE CASE OF GEOMETRY

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Looking at Figure 1a, I say duck, you say rabbit, so let's call the whole thing off, because it can't be both. Looking at Figure 1b, though, I say two rows of three X's, you say three columns of two X's, so let's not call the whole thing off, because our disagreement could be reconciled in the form of a mutually valuable insight into the commutative property of multiplication, where the two perceptual orientations are complementary construals of six X's (*i.e.,* $2 \times 3 = 3 \times 2$). Abrahamson and Wilensky (2007) used this example to introduce an educational design framework—*learning axes and bridging tools*—centered on fostering conceptual insight through setting up students to experience then reconcile ambiguous perceptual constructions of instructional materials. Engaging with these materials, students are to experience different meanings that are each valid in their own right yet initially appear incompatible with each other. The learning goal requires finding a new way of thinking that would accommodate or resolve the conflict, whereby the alternative perceptions become complementary or dialectic rather than contradictory.

The educational design principle of learning through reconciling competing perceptual constructions has been applied also to the case of ratio and proportion (*e.g.,* Abrahamson, Lee, Negrete & Gutiérrez, 2014). The objective of the current article is to investigate the application of the framework to geometry, in particular to designing activities where students engage in task-oriented embodied investigations into voluminous objects. The idea is that students build these objects themselves, moving from 2D images to 3D structures. These objects, built at different scales, offer multiple situated perspectives and, hence, opportunities for richer and ecologically authentic collaborative sense-making (see Figure 1c). In so doing, we are also interested in further theorizing, elaborating, and refining the design framework. In particular, we wish to foreground the formative role of perception in our design and analysis of learning opportunities. To these ends, we introduce the notion of *conceptually generative perspectival complementarities* (CGPC), that is, pairs of practical orientations toward task materials, where each way of perceiving is contextually valid. We posit that by coordinating these seemingly incompatible construals students can gain conceptual insight into the content. Our goal is to understand the socio-cognitive micro-process by which CGPC designed by researchers are animated by collaborating students in ways that bring about learning opportunities.

The article surveys a set of three design-based research projects centered on different notions related to geometry and spanning elementary-, middle-, and high school. Looking at

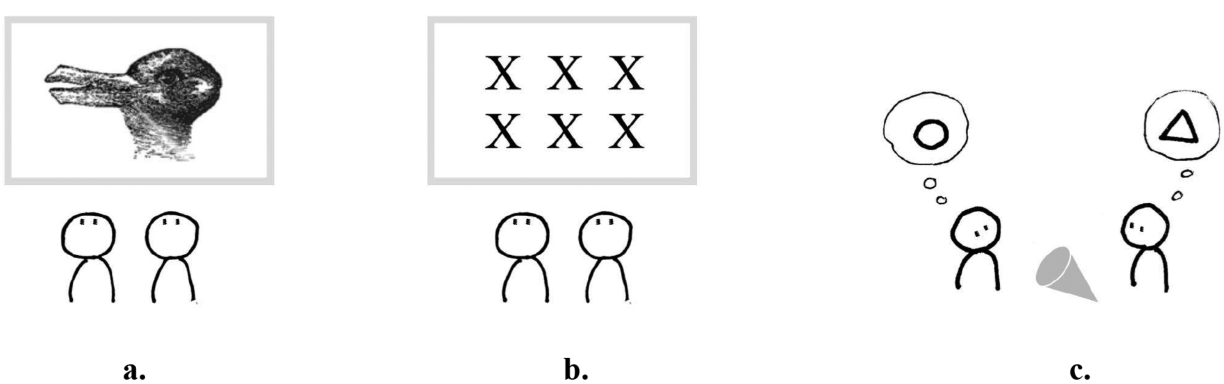

Figure 1. On ambiguity, perception, and learning: (a) Jastrow's duck/rabbit; (b) An array of six Xs. (c) A geometric solid. As we view a display, what might we learn from comparing its competing perceptual constructions?

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pilot empirical data from these projects, we will address the following questions:

> What CGPC did the activity occasion and leverage?
>
> How did the socio-material task configuration occasion the CGPC?
>
> What are students' apparent discursive practices for negotiating CGPC?
>
> What are the epistemic consequences of dialogic negotiations among the students: Did tackling CGPC together result in a stubborn 'duck,' a more expansive 'duck OR rabbit,' or a conceptually edifying 'duck–rabbit' [1]?

Before looking at the projects, we will further motivate the framework by way of foregrounding the constitutive role of perception in developing mathematical understandings as well as the formative role of interpersonal coordination in forging new perceptual orientations to shared situations.

Grounding mathematical concepts in negotiated perceptual constructions of learning materials

When we consider what students might learn through engaging with educational artifacts, a central theoretical construct guiding our inquiry is that of perception. Similar to radical constructivists (Steffe & Kieren, 1994), we think of perception as cognitive activity. Perception is an individual's mental construction of the environment guiding their action-oriented sensorimotor immersion at a given moment in the context of some activity task. As in the paradigmatic case of Jastrow's duck/rabbit, a student may perceive the same features differently from one moment to the next, or two students may perceive the features differently at the same moment. In turn, opportunities for action that students experience as they engage with educational materials depend on what they already know to do, and new opportunities for action then emerge through handling the materials and reflecting on surprising outcomes, which forge new skills and knowledge (Roth, 2010).

While perception *organizes* interaction, it *focalizes* discourse. We talk about what we know to do, what we are trying to do, and how we might do it together. Disciplinary language introduced into such discussions about artifacts tends to form and reconfigure our perception of the artifacts (Bartolini Bussi & Mariotti, 2008). In particular, when students coordinate their interactions with artifacts, either among themselves or with teachers, ambiguities may lurk under their actual or ostensible mutual understanding, because their respective perceptions of the artifacts may differ. Far from causing impasses, these ambiguities are desirable, because they enable and mobilize reflection, insight, and enculturation (Newman, Griffin & Cole, 1989). Objectively, it is the same artifact, but subjectively it is now construed anew so as to combine CGPC to engage productively in the classroom's mathematical activity.

Your team has to construct two three-dimensional models (one large, one small) of a geometric solid, a polyhedron.

The polyhedron has the following properties:

- All the faces are congruent equilateral triangles.
- The same number of edges converge at each vertex.

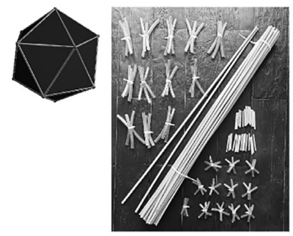


Figure 2. The icosahedron construction task and materials. Left: Worksheet for student teams; Right: The construction kit, with long dowels for the large form, small dowels for the small form and two sets of 12 silicone joints to serve as vertices.

Three vignettes

Our project follows a pedagogical conjecture that *geometry learning activities should engage students in purposive collaborative tasks to solicit conversations around their perceptual orientation to the materials.* The project is distributed across three contexts, with diverse populations. Our preliminary empirical data so far come from: (1) *Inside Geometry, and Out!* (Jerusalem, Israel): 60 high-school students in an enrichment activity and 9 university students in the course 'New Ways to Think, Learn, and Move'; (2) *VR SandScape* (Oakland, California, USA): 16 middle school students from a multicultural urban secondary school; and (3) *Indigeometry Planetarium* (Tohatchi, New Mexico, USA): 1 Navajo student. All contexts were designed to bring forth epistemic, affective, and social CGPC.

Vignette 1

The empirical context of *Inside Geometry, and Out!* explores collaborative learning about geometric solids. A distinctive feature of this environment is that students construct the same geometric objects at different scales using wooden rods and silicone joints. In one of the tasks (see Figure 2) students explore an icosahedron. In this activity, students are given a 2D-diagram and are to construct a relatively small icosahedron as well as a human-scale icosahedron. Once both models are built, students are asked questions concerning the icosahedron's geometric and topological properties, for example, 'How many vertices does an icosahedron have?', 'How many parallel edges?', 'If the icosahedron were standing on its triangular base and filled half-way up with water, what would be the water's surface shape?' We analyze students' choices of small versus large icosahedra to investigate each question.

The following account is from a pilot outdoor implementation of the activity with a group of high-school students. It describes how they worked with their constructed models to solve the questions (see Figure 3).

Having constructed both the large and small models, the students used the small model to answer correctly that an icosahedron has 12 vertices. Next, they tackled the question of how many edges an icosahedron has. They soon found it difficult to solve this problem using the small model [2].

Yellow	How many edges are there?
Black	Okay, that's tricky because they're

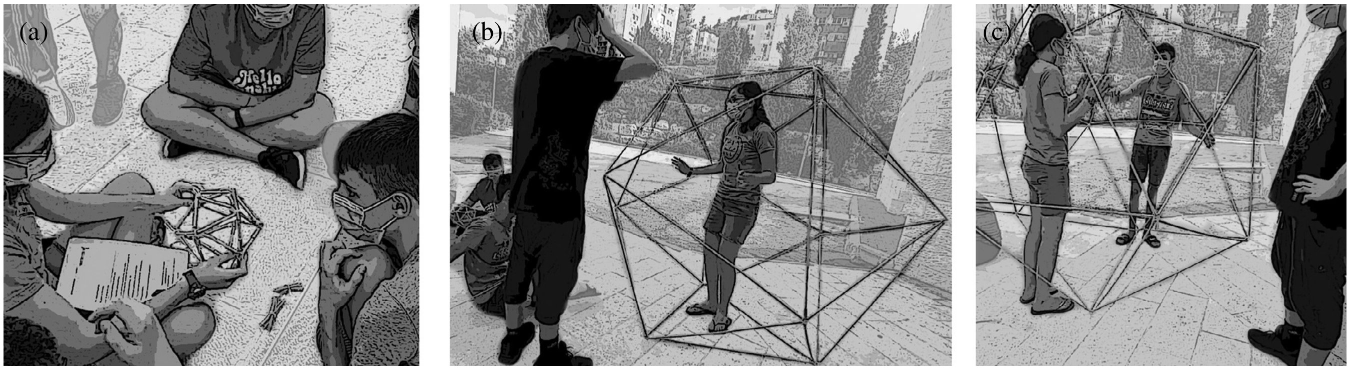


Figure 3. The geometry activity Inside Geometry, and Out! combines construction, problem-solving, and justification tasks, where each task provides different CGPC-related coordination challenges. (a) Students discuss a small-scale model; (b) Students' problem-solving inside and outside a human-scale model (standing on a triangular face); (c) Having tilted the structure onto a vertex, the students soon arrive at a critical breakthrough.

shared. *[each edge is shared by two triangles].*

Blue — I'll put a finger *[on the first edge that Yellow counts, to help her monitor the count].*

Orange — You just count the sticks.

Yellow — I'll go to the big one *[the large-scale model].*

Black — The big one is just nicer.

Three of the six students rose and walked over to the large-scale model. This larger model is advantageous for counting, because its edges are more perceptually distinct. But the model's greater size, while aiding perception, may come at a price. Its elements (*e.g.,* the to-be-counted edges) are never all in reach—you cannot directly touch or clearly gesture to each edge as you tally it. Immediately, Yellow went inside the model, which put all the edges within her reach. Still, from inside an object, part of it is always behind you, and so you might lose track of your count! Indeed, Yellow's initial attempts to count failed. As the excerpt below demonstrates, she then attempted to use some of the icosahedron properties that the team had discovered during construction, yet again she failed to develop a systematic approach.

Yellow — There are five from each vertex. One should be subtracted, then there are four. Two should be subtracted here, it's three. It doesn't work that way. *[pause]* 3, 4, 5. I can't count this. How many sticks did we use *[during the construction stage]*? Three and another three, and another three, and another three, and another three, it's 12, another three, 15, another three *[referring to triangular faces] [pause]*

Black — We need a formula for this.

From a mathematical point of view, it does not matter how the icosahedron is positioned in space—the polyhedron's topological properties remain the same. In a gravitational world, however, the model usually lies on one of its triangular faces, making it difficult to perceive certain structural symmetries. The next excerpt shows how, by tilting the model onto a vertex (see Figure 3c), suddenly these inherent symmetries became apparent: two opposing 'bases,' each comprising 10 edges, and a connecting 'belt,' also with 10 edges.

Gray — It will be easier to count like that *[tilts the model so it stands on a vertex and holds the model in place].* 1, 2, 3, 4, 5 *[counts the edges diverging from the base vertex]*; 1, 2, 3, 4, 5 *[counts the edges of the pentagonal base]*

Yellow — 1, 2, 3, 4 *[pause]* 1, 2, 3, 4 *[pause] [addressing Grey]* Put your hand here. 1, 2, 3, 4, 5 *[continues to count silently].* Ten, ten, ten *[pause]* thirty!

In sum, the activity of exploring a geometric object by constructing it at different physical sizes surfaced and leveraged two CGPC: scalar (large model versus small) and situated (inside versus outside the model). Several features of the socio–material task configuration combined to draw out and mobilize these CGPC.

First, each model served as a physical attractor with different affordances for, and constraints on action; accordingly, the group of students reorganized spontaneously around these affordances and constraints. For instance, the small-scale model centered the group's interactions, but its modest size could not accommodate their desired forms of inquiry; the availability of a larger model catalyzed splitting the group, which, in turn, juggled the students' social roles.

Second, achieving the sub-goal of counting the edges required one of the students to mobilize group resources (physical, perspectival, intellectual). In negotiating emergent perspectival complementarities, the students combined collaborative action, gesture (indexing and iconic), and speech to indicate and highlight for each other the model's figural properties. Tilting the model onto a vertex helped one of the students share his perspective with his teammates, who could then perceive how he was parsing the structure and could, therefore, count up the edges using this structural insight. The fluency with which students moved from one model to another—both physically and inferentially—suggests they were noticing invariant scale-free features of a geometric object. A multitude of perspectives converged to generate the concept of the polyhedron.

Vignette 2

VR SandScape is a hybrid Spatial Augmented Reality (SAR) sandbox and Virtual Reality (VR) system that we developed to support children's collaborative design processes involving geometrical solids and topographic projections. Using a depth-sensing camera installed above the physical sandbox, the system scans the surface of the sand in real time and generates a correlated 3D, VR rendering of the sandbox topology that is constantly changing as one child physically sculpts the 'sandscape' (see Figure 4). In the corresponding VR world, a second child wearing a head-mounted display (HMD) can virtually walk through the mountains, valleys, *etc.* that were physically crafted in the sandbox, with a first-person point of view. We intentionally employ only one HMD, as we want children to take turns being the physical landscape manipulator at the sandbox and the immersed explorer in the VR world. Thus, one child could see the terrain from an 'outside' and broader perspective while another child experiences the same terrain in VR from an 'inside' perspective. For example, a child in VR might suddenly see a gaping canyon appear in front of her because her partner in the physical world just scooped up a handful of sand. The sandbox is augmented with color projections from above to visually emphasize topography such as lakes, peaks, *etc.* The virtual model uses the same colors as the projection.

In our preliminary study, we asked middle school children to work in pairs to design a maze that has three mountains to climb anywhere along the path, where Mountain A must be two times taller than Mountain C, Mountain B has to be three times taller than Mountain C, and Mountain C can be any height.

In the following example, Sam wears a HMD and explores the model in VR while Ruth physically sandscapes the maze. A nearby screen shows Sam's view.

Sam — [*talking to Ruth*] However we are trying, just make sure that, it's, um, big enough to be considered a mountain. And small enough to [*pause*] make sure it could be three times as large as *[pause]* for the mountain B.

Ruth — Uh huh. I'm also making a path while I'm making this mountain.

[Later]

Sam — Where do you think we should put the other mountains? *[looking around in VR]*

Sam — Do you see where I am looking?

Ruth — Yeah. *[Ruth goes back and forth between Sam's perspective provided by the screen and her own perspective of the physical sandbox]*

Ruth — I think we should put them a fair distance apart. So that they are not clumped up in one location.

Figure 4. Ruth is measuring the height of the tall mountain using a physical ruler, while her partner Sam looks at the model from the VR perspective.

Sam	Do you see where I am looking? Somewhere over there? *[speaking from her VR perspective]*
Ruth	Here. *[points to a location in the sandbox. Now Ruth's hand is represented as a part of landscape Sam can see in VR]*
Sam	Right there? OK. *[Sam sees what Ruth is pointing at as a part of landscape in her VR view]*
Sam	Um, maybe Mountain B there.
Ruth	Mountain B goes here. *[Ruth is looking at the sandbox and screen, and starts sculpting Mountain B]*
Sam	Yeah. So the one that's three times as big as this one. *[speaking from the VR view]*

Both Ruth and Sam communicate their effort to fulfill the task requirements as well as aesthetic concerns for their maze design. Practically, it would be simpler to have the mountains close to each other so that the heights could be compared easily, yet Ruth wishes the mountains to be a "fair distance apart" for aesthetic reasons. Likewise, making Mountain C small would keep Mountain B's height manageable. Sam communicates this point to Ruth while also assuring that it should be "big enough to be considered a mountain" from the perspective of an actual maze user. Through accessing the multiple views available to them (*i.e.,* view of physical sandbox, VR view, view of each other), Ruth and Sam negotiate their respective perspectives, being inside versus outside the model (situated CGPS) and set up their own goals in achieving a design that meets the task specifications.

[Ruth is at the sandbox measuring their small mountain with a physical ruler (see Figure 4, on the left).]

Ruth	Two and a half inches. *[Ruth reads the ruler.]* So then, like *[pause]* around seven inches. *[Ruth looks at the ruler, and now measures the taller mountain]* OK. Cool. *[Ruth recognizes that the taller mountain is not tall enough. She then puts the ruler away and makes the tall mountain even taller.]*
Sam	I just saw the mountain. *[laughs] [Sam is seeing the mountain being created by Ruth in the VR perspective.]*
Ruth	This mountain is really steep. *[Ruth finishes the tall mountain in the sandbox.]*
Ruth	There we go. *[Ruth takes a physical ruler and measures the height of the tall mountain she just created in the sandbox.]*
Ruth	Yup. Still not tall enough. Alright. *[Ruth puts the ruler away and sculpts the mountain to be even taller.]*
Sam	Remember, you can make the other one shorter. *[laughs] [Sam in VR view]*
Ruth	Duh! *[laughs]*

In sculpting sand mountains physically, Ruth struggles to make Mountain B three times taller than Mountain C. Sam, with her VR HMD, has an 'inside' perspective on the sculpted mountains, which enables her to notice and communicate with her partner that Mountain C could be made smaller to facilitate the construction of Mountain B. As such, the two children take turns being the creator of the physical model and the evaluator of the same model from the VR perspective, collaborating simultaneously at two different scales, surfacing scalar CGPC. It is this turn-taking between the two roles, creating and evaluating a shared arena from multiple perspectives, which provides an opportunity for dialogic reconciliation of these different scalar views. This, in turn, could surface conceptually productive differences relevant to the study of geometrical solids and topographic projections.

Ruth	*[Ruth is at the sandbox creating steps with a ruler.]* So the end of mountain A. We are gonna put steps. Steps, steps.
Ruth	*[talking to Sam]* I'm gonna need you to come over here to see if these steps are way too tall.

[Sam comes towards Mountain A in the virtual world.]

Sam	OK I'm at Mountain A. Pretty much, where *[pause]*
Ruth	You are right at the steps.
Sam	OK, they are not really defined.
Ruth	I'm just gonna make like railings here. *[Ruth makes rails using rulers.]* Perfect.
Sam	OK Let's make some cliffs, right?
Ruth	Yeah.
Sam	Maybe have one cliff, the one that goes, and up! To like, right here? But then, there, it's just kind of cut off but then there's a railing type of thing so it doesn't knock the person off the cliff. Yeah.

Ruth asks Sam in the VR to come towards the steps Ruth just created and evaluate them from Sam's perspective as a user. Here, Ruth temporarily replaces her perspective (as a designer and builder of the maze) with Sam's (a maze user) as it is contextually advantageous for her to gain information relevant to building the maze. Later, as Sam continues to co-design the maze with Ruth, Sam refers to "the person" who is different from themselves. Both Ruth and Sam are thinking prospectively about the 'future user' of the maze, which demonstrates a *synergy* of a new perspective that is greater than the sum of their respective views.

Vignette 3

Indigeometry Planetarium is a learning environment (Figure 5) constructed as a canvas-covered dome. In the planetarium, students enact essential perspectival qualities of Navajo archaeoastronomical practice in negotiation with Euclidean geometry. Navajo perspectival qualities are enmeshed in the Navajo nation's cultural–historical practice that, in turn, is rooted in the unique topography, climate, flora, and fauna of Navajo land. In particular, Navajo archaeoastronomy is predicated on human-centric perception of the earth's orbit vis-à-vis the open starry skies and familiar mountains. Relatedly, the epistemological foundation of the Navajo language —the very cognitive morphology of its grammatical constructions—intrinsically places individuals as active participants in their perceptual observations, reflecting a phenomenologically irreducible systemic connection between mind, body, and land; a perspective that differs from Western (*i.e.* colonial) allocentric view (Barton, 2008). As such, Navajo perceptual experience of an angle is inclusive, where the rays extend 'ecogenetically,' that is, from the boundaries of the observed object toward the viewer-as-vertex of an egocentric perspectival triangle. By virtue of looking up to the stars, a Navajo astronomer thus participates in a triangular perspectival structure: human perception subtends the celestial percept—say, an imaginary line connecting two stars—the observer, thus, intrinsically *becomes* or *incorporates* the angle (compare to Gerofsky, 2011).

Amaya, an 8 year-old 3rd-grade female Navajo student, participated in a 30-minute semi-structured task-based interview conducted by Jessica (the first author). The first part of the activity is designed to introduce an egocentric notion of angle: the student, seated on a swivel chair in the center of the planetarium, becomes the vertex of an angle that increases through angular rotation of the body. Jessica asked Amaya, "Can you point at the *start* of the shooting star?" Amaya raised her left hand and pointed to the left-side end of a shooting star ahead and above her. Further instructed to "use the other hand to point at the *end* of the shooting star", Amaya, still holding her left hand up, raised her right hand and pointed to the right-side end of the same star. Her two arm–rays now projected from her body–origin to these stars to embrace their distal span (see Figure 5). This initial activity established the planetarium as a field of promoted action, wherein the student enacted a conceptual choreography—opening and closing her arms—that will prospectively ground the targeted concept of angle (Abrahamson, 2019).

The next activities are designed to bring the interior of the rotating arms into focus and formalize it as a static, measurable angular magnitude. First, students distinguish the plane in which their arms rotate by performing a full rotation. Next, this full rotation is associated with a 360 degree measure. Finally, students are guided to perceive their rotating arms as dynamically marking an angular measure.

The activity started when Jessica instructed Amaya to extend her arms out to her sides and turn around on the spot, completing a full rotation. Jessica asked Amaya, "Can you feel what shape your arms made?" Amaya answered, "circle". Jessica then gestured the full circle and called it 360 degrees. Next, she presented half a circle as "half of 360 degrees, which is 180", while she simultaneously extended her arms forward together then opened them to the sides. This action is meant to present the interior angular expansion in relation to the entire sweep.

For the last activity, the arms–protractor was introduced to refine the quantification of angularity. After Amaya inspected the arms–protractor (Figures 6a, b), Jessica demonstrated how to use it: "You hold it to your chest like this [tool joint is by the chest, each hand holding a dowel near the joint]—these will be your arms now". Amaya, holding the tool, placed her hands close to the joint, similar to Jessica. She pointed one dowel to one side of the shooting star, then pointed the other dowel to the other side of the star (Figure 6c). Once the measure was fixed, Jessica showed Amaya how "to see what the number is" (Figure 6d). Amaya said, "40", and Jessica agreed, "Yeah [giving her back the tool], this shooting star from where you are sitting is 40 degrees apart".

Figure 5. Embodied measuring in the Indigeometry Planetarium. (left) The icosahedron-shaped dome is constructed from dowels and plastic tube joints and covered with black nylon material. The inside ceiling features appliqué stars. (right) Gazing toward a shooting star, Amaya's left- and right-hand index fingers are pointing respectively to its left- and right sides. Jessica looks on.

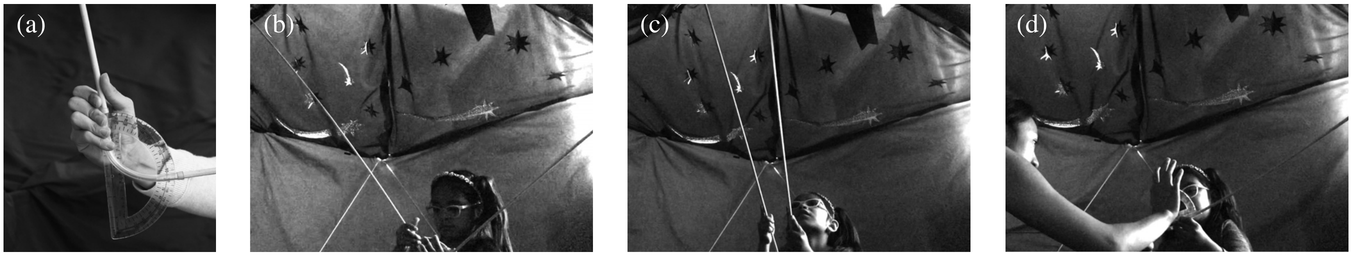


Figure 6. *Arms–protractor tool. (a) A close-up view of the arms–protractor joint. (b) Amaya examining the arms–protractor. (c) Amaya uses the tool to qualitatively measure the distance between her hands when pointing to a shooting star, which represents the multiplicative comparisons of circular arc lengths. (d) Jessica demonstrating how to read the numerical angle measure from the protractor.*

Jessica then assigned Amaya the task of measuring all the shooting stars with the arms–protractor.

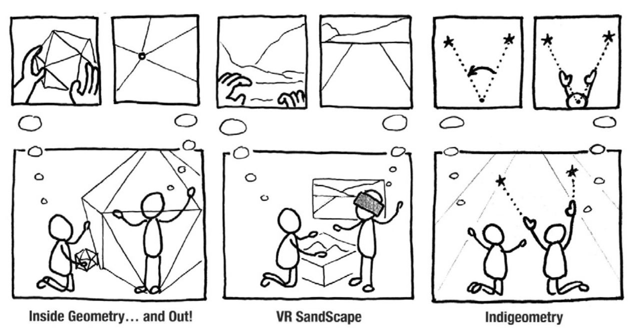
The vignette demonstrated an embodied, student-centered, multimodal approach to the teaching of angles (Smith, King & Hoyte, 2014). The activity staged a cultural CGPC by combining the Navajo perspective on angle (viewer of the stars becomes an angle) and the Euclidean perspective (viewer away from the angle). These culturally distinct perspectives become coordinated by means of the arms–protractor, which constitutes a *bridging tool,* in the sense that its structure and suggested use elicit essential elements of two different perceptual practices that are phenomenologically disparate yet conceptually complementary (Abrahamson & Wilensky, 2007). Specifically, the arms–protractor was designed to offer a perspectival complementarity that would ground Euclidean quantification of angle in Navajo perceptual practice. As shown in Figure 6a the arms–protractor incorporates a half-disk protractor. This allows the angle made by the arms to be related to a numerical magnitude. As we analyze students' multimodal behaviors, we look for indications that they are coordinating the CGPC as shown by gestures and words referring to both enactive and quantitative facets of the activity. In that sense, we search for critical learning events that Bartolini Bussi and Mariotti (2008) might recognize as 'pivot signs' connecting artifact signs and mathematical signs.

We point out three design principles relating to multimodality that facilitated Amaya's negotiation of a cultural CGPC. First, enacting a single posture with two distinct attentional foci yet potentially complementary perceptual meanings (standing with arms reaching to a star as either a Navajo measure of aperture-as-span or a Euclidean measure of angle-as-rotation at the vertex). Second, introducing mathematical instruments as extending naturalistic actions (the arms–protractor extending the student's pointing toward objects of interest). Finally, legitimizing multimodal expressions in mathematical argumentation (invoking embodied actions to support conceptual explanation; see also Feucht, 2010, on expanding classroom epistemic climate). Future iterations of the *Indigeometry Planetarium* activity design will encompass student–student collaboration.

Substitution, mutuality, and synergy

The three studies above exemplify our collective efforts to develop an explanatory process model of geometry learning, as learners' negotiation across different perceptual perspectives on a given situation (see Figure 7). The vignettes suggest the validity and robustness of our thematic construct, conceptually generative perspectival complementarities (CGPC), by demonstrating similar interaction patterns across variable orders of participation—group (or pair; inter-personal negotiation) and individual (intra-personal negotiation). In all three studies, the activities were designed to foster differing perceptual orientations toward the material objects—differing perceptual orientations whose recognition and reconciliation promoted conceptual understanding. In *Inside Geometry, and Out!,* students simultaneously and collaboratively examined an icosahedron from within and without. In *VR SandScape,* two students who experienced a topology on the ground and from a bird's-eye view, respectively, built shared verbal references. In *Indigeometry Planetarium*, the child coordinated her allocentric experience of a static angle with her egocentric dynamic rotational experience, enabling her to orient dynamically–quantitatively to the static image. In all studies, the construct of CGPC illuminated the opportunities inherent in the design as they played out in its implementation.

Across the vignettes, the discursive negotiation of perspectives played a crucial role in mobilizing the participants' actions and insights. The collaborating children were engaged in pragmatic discourse negotiating different



(a) (b) (c)

Figure 7. *The project's three learning environments viewed as perceptual complementarities. (a) External versus internal views of a body-scale geometric form; (b) Top-down versus immersed views of an adventure landscape; (c) Allocentric versus egocentric views of a planetarium star constellation.*

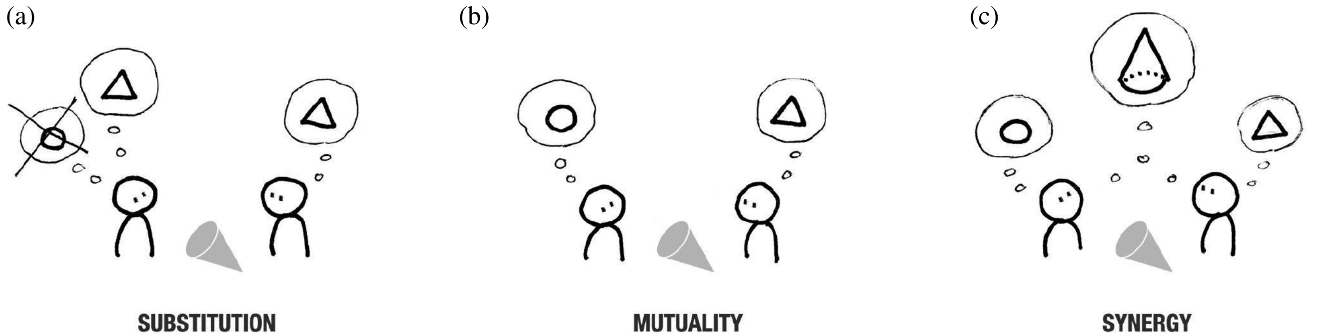


Figure 8. *Consider two persons observing one geometric object from two different points of view. (a) One of them sees a circle and the other a triangle. For some reason, they may decide that they are interested only in the triangular properties of the object. (b) They also may be satisfied with their new awareness of a different facet of the object. (c) They also may realize that the object in question is a cone.*

perspectives on the activity's focal objects. Not only did the students surface and reify the objects' geometric properties from their subjective perspective—they also identified and 'conserved' a set of pan-perspective, scale-free, and, thus, mutually intelligible features, according to the design's objectives. For instance, in the first vignette the number of edges does not depend on the size of a model and its position in space. In the second, the ratio of two mountains' heights does not depend on their perceived size. In the third, the amount of turn defines an angle and does not depend on the Navajo or Western perspective.

Comparative analysis of the vignettes suggests a new elaboration of CPGC by negotiation *type:* substitution, mutuality, and synergy (see Figure 8). In *substitution*, one of the perspectives replaces the other, because it is tacitly evaluated as contextually advantageous for attaining information relevant to the task at hand (*e.g.,* when Amaya utilized arm aperture as a qualitative measuring tool). In *mutuality*, the viewing perspectives are both sustained, with participants retaining their initial perspective while sanctioning the alternative view (*e.g.,* when Sam and Ruth each kept experiencing the landscape from their respective scale). In *synergy*, a new perspective emerges that is greater than the sum of its parts (*e.g.,* counting together the edges of an icosahedron from within and without it).

Conclusion

When students have different perspectival orientations on a shared task, this difference can be leveraged as a means of transitioning from intuitive to disciplinary practices and understandings. As they figure out together how best to collaborate across their perspectival differences, students' perspectival transaction results either in substitution, mutuality, or synergy, with varying consequences for learning outcomes. We are only beginning to identify which sociomaterial circumstances result in each form of negotiation, and how these negotiation outcomes bear on the emergence of mathematical ontologies. As educational designers, we look to understand the role of different media in facilitating productive negotiations, and we seek to investigate challenges and opportunities of integrating CGPC activities into classroom settings and mainstream curriculum.

More broadly, we aspire to delineate heuristics for creating activities that optimize for learning across perspectival differences. Ultimately, an approach to the learning of mathematics grounded in reconciling perspectival complementarities, we surmise, could bear on broader ideological and socio-political issues of diversity, inclusiveness, and cultural identity.

Notes

[1] We use an en-dash throughout to combine words of equal status.

[2] Participants are referred to by the color of their t-shirts; transcription translated from Hebrew by AP.

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